Exercise 4.1

Question 1:

Evaluate the determinants in Exercise 1 and 2. $\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix}$

Solution 1:

$\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix} = 2(-1) - 4(-5) = -2 + 20 = 18$

Question 2:

Evaluate the determinants in Exercise 1 and 2.

(i) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$ (ii) $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$

Solution 2:

(i) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = (\cos\theta)(\cos\theta) - (-\sin\theta)(\sin\theta) = \cos^2\theta + \sin^2\theta = 1$

(ii) $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$

$= (x^2 - x + 1)(x + 1) - (x - 1)(x + 1)$

$= x^3 - x^2 + x + x^2 - x + 1 - (x^2 - 1)$

$= x^3 + 1 - x^2 + 1$

$= x^3 - x^2 + 2$

Question 3:

If $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$, then show that $|2A| = 4|A|$

Solution 3:

The given matrix is $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$

$\therefore 2A = 2\begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 8 & 4 \end{bmatrix}$

$L.H.S : |2A| = \begin{vmatrix} 2 & 4 \\ 8 & 4 \end{vmatrix}$

$= 2\times4 - 4\times8$

$= 8 - 32 = -24$

Now, $|A| = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix} = 1\times2 - 2\times4 = 2\times8 = -6$

$R.H.S: 4|A| = 4\times(-6) = -24$

$\therefore L.H.S. = \therefore$R.H.S.

Questions 4:

$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$, then show that $|3A| = 27|A|$.

Solution 4:

The given matrix is $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$.

It can be observed that in the first column, two entries are zero. Thus, we expand along the first column (C_1) for easier calculation.

$|A| = 1\begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} - 0\begin{vmatrix} 0 & 1 \\ 0 & 4 \end{vmatrix} + 0\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = 1(4-0) - 0 + 0 = 4$

$\therefore 27|A| = 27(4) = 108$ ……(i)

Now, $3A = 3\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 12 \end{bmatrix}$

$\therefore |3A| = 3\begin{vmatrix} 3 & 6 \\ 0 & 12 \end{vmatrix} - 0\begin{vmatrix} 0 & 3 \\ 0 & 12 \end{vmatrix} + 0\begin{vmatrix} 0 & 3 \\ 3 & 6 \end{vmatrix}$

$= 3(36-0) = 3(36) = 108$ ……(ii)

From equations (i) and (ii), we have:

$|3A| = 27|A|$

Hence, the given result is proved.

Question 5:

Evaluate the determinants

(i) $\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$ (ii) $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$

(iii) $\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix}$ (iv) $\begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

Solution 5:

(i) let $A = \begin{bmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix}$

It can be observed that in the second row, two entries are zero.
Thus, we expand along the second row for easier calculation.

$$|A| = -0\begin{vmatrix} -1 & -2 \\ -5 & 0 \end{vmatrix} + 0\begin{vmatrix} 3 & -2 \\ 3 & 0 \end{vmatrix} - (-1)\begin{vmatrix} 3 & -1 \\ 3 & -5 \end{vmatrix} = (-15+3) = -12$$

(ii) Let $A = \begin{bmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{bmatrix}$

By expanding along the first row, we have:

$$|A| = 3\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} + 4\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} + 5\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}$$

$$= 3(1+6) + 4(1+4) + 5(3-2)$$

$$= 3(7) + 4(5) + 5(1)$$

$$= 21 + 20 + 5 = 46$$

(iii) Let $A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix}$

By expanding along the first row, we have:

$$|A| = 0\begin{vmatrix} 0 & -3 \\ 3 & 0 \end{vmatrix} - 1\begin{vmatrix} -1 & -3 \\ -2 & 0 \end{vmatrix} + 2\begin{vmatrix} -1 & 0 \\ -2 & 3 \end{vmatrix}$$

$$= 0 - 1(0-6) + 2(-3-0)$$

$$= -1(-6) + 2(-3)$$

$$= 6 - 6 = 0$$

(iv) Let $A = \begin{bmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{bmatrix}$

By expanding along the first column, we have:

$$|A| = 2\begin{vmatrix} 2 & -1 \\ -5 & 0 \end{vmatrix} - 0\begin{vmatrix} -1 & -2 \\ -5 & 0 \end{vmatrix} + 3\begin{vmatrix} -1 & -2 \\ 2 & -1 \end{vmatrix}$$

$$= 2(0-5) - 0 + 3(1+4)$$

$$= -10 + 15 = 5$$

Question 6:

If $A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{bmatrix}$, find $|A|$

Solution 6:

Let $A=\begin{bmatrix}1 & 1 & -2\\ 2 & 1 & -3\\ 5 & 4 & -9\end{bmatrix}$.

By expanding along the first row, we have:

$$A=\begin{bmatrix}1 & 1 & -2\\ 2 & 1 & -3\\ 5 & 4 & -9\end{bmatrix}$$

$$\begin{aligned}|A| &= 1\begin{vmatrix}1 & -3\\ 4 & -9\end{vmatrix}-1\begin{vmatrix}2 & -3\\ 5 & -9\end{vmatrix}-2\begin{vmatrix}2 & 1\\ 5 & 4\end{vmatrix}\\ &=1(-9+12)-1(-18+15)-2(8-5)\\ &=1(3)-1(-3)-2(3)\\ &=3+3-6\\ &=6-6\\ &=0\end{aligned}$$

Question 7:

Find values of x, if

(i) $\begin{vmatrix}2 & 4\\ 5 & 1\end{vmatrix}=\begin{vmatrix}2x & 4\\ 6 & x\end{vmatrix}$ **(ii)** $\begin{vmatrix}2 & 3\\ 4 & 5\end{vmatrix}=\begin{vmatrix}x & 3\\ 2x & 5\end{vmatrix}$

Solution 7:

(i) $\begin{vmatrix}2 & 4\\ 5 & 1\end{vmatrix}=\begin{vmatrix}2x & 4\\ 6 & x\end{vmatrix}$

$$\Rightarrow 2\times1-5\times4=2x\times x-6\times4$$
$$\Rightarrow 2-20=2x^2-24$$
$$\Rightarrow 2x^2=6$$
$$\Rightarrow x^2=3$$
$$\Rightarrow x=\pm\sqrt{3}$$

(ii) $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

$\Rightarrow 2\times5-3\times4=x\times5-3\times2x$
$\Rightarrow 10-12=5x-6x$
$\Rightarrow -2=-x$
$\Rightarrow x=2$

Question 8:

If $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$, then x is equal to

A. 6
B. ±6
C. -6
D. 0

Solution 8:

$\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$

$\Rightarrow x^2-36=36-36$
$\Rightarrow x^2-36=0$
$\Rightarrow x^2=36$
$\Rightarrow x=\pm6$

Hence, the correct answer is B.

Exercise 4.2

Question 1:

Using the property of determinants and without expanding, prove that:

$$\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = 0$$

Solution 1:

$$\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = \begin{vmatrix} x & a & x \\ y & b & y \\ z & c & z \end{vmatrix} + \begin{vmatrix} x & a & a \\ y & b & b \\ z & c & c \end{vmatrix}$$

Clearly, the two determinants have two identical columns. Thus,

$$= 0 + 0 = 0$$

Question 2:

Using the property of determinants and without expanding, prove that:

$$\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$$

Solution 2:

$$\Delta = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2$, we have:

$$\Delta = \begin{vmatrix} a-c & b-a & c-b \\ b-c & c-a & a-b \\ -(a-c) & -(b-a) & -(c-b) \end{vmatrix}$$

$$= -\begin{vmatrix} a-c & b-a & c-b \\ b-c & c-a & a-b \\ a-c & b-a & c-b \end{vmatrix}$$

Here, the two rows R_1 and R_3 are identical.

$\therefore \Delta = 0.$

Question 3:

Using the property of determinants and without expanding, prove that:

$$\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} = 0$$

Solution 3:

$$\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} = \begin{vmatrix} 2 & 7 & 63+2 \\ 3 & 8 & 72+3 \\ 5 & 9 & 81+5 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 7 & 63 \\ 3 & 8 & 72 \\ 5 & 9 & 81 \end{vmatrix} + \begin{vmatrix} 2 & 7 & 2 \\ 3 & 8 & 3 \\ 5 & 9 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 7 & 9(7) \\ 3 & 8 & 9(8) \\ 5 & 9 & 9(9) \end{vmatrix} + 0 \qquad [\text{Two Coloumns are identical}]$$

$$= 9\begin{vmatrix} 2 & 7 & 7 \\ 3 & 8 & 8 \\ 5 & 9 & 9 \end{vmatrix}$$

$$= 0 \qquad [\text{Two Coloumns are identical}]$$

Question 4:

Using the property of determinants and without expanding, prove that:

$$\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0$$

Solution 4:

$$\Delta = \begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}$$

By applying $C_3 \to C_3 + C_2$. We have:

$$\Delta = \begin{vmatrix} 1 & bc & ab+bc+ca \\ 1 & ca & ab+bc+ca \\ 1 & ab & ab+bc+ca \end{vmatrix}$$

Here. Two columns C_1 and C_3 are proportional.

$\therefore \Delta = 0.$

Question 5:

Using the property of determinants and without expanding, prove that:

$$\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} = 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

Solution 5:

$$\Delta = \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & z+y \end{vmatrix}$$

$$= \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a & p & x \end{vmatrix} + \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ b & q & y \end{vmatrix}$$

$$= \Delta_1 + \Delta_2 \ (\text{say}) \qquad \text{.......(1)}$$

Now, $\Delta_1 = \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a & p & x \end{vmatrix}$

Applying $R_1 \to R_1 - R_2$, we have:

$$\Delta_1 = \begin{vmatrix} b & q & y \\ c & r & z \\ a & p & x \end{vmatrix}$$

Applying $R_1 \leftrightarrow R_3$ and $R_2 \leftrightarrow R_3$, we have:

$$\Delta_1 = (-1)^2 \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} \qquad \text{...... (2)}$$

$$\Delta_2 = \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ b & q & y \end{vmatrix}$$

Applying $R_1 \to R_1 - R_3$, we have:

$$\Delta_2 = \begin{vmatrix} c & r & z \\ c+a & r+p & z+x \\ b & q & y \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$, we have:

$$\Delta_2 = \begin{vmatrix} c & r & z \\ a & p & x \\ b & q & y \end{vmatrix}$$

Applying $R_1 \leftrightarrow R_2$ and $R_2 \leftrightarrow R_3$, we have:

$$\Delta_2 = (-1)^2 \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} \qquad \ldots(3)$$

From (1),(2), and (3), we have:

$$\Delta = 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

Hence, the given result is proved.

Question 6:

By using properties of determinants, show that:

$$\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0$$

Solution 6:

We have,

$$\Delta = \begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$$

Applying $R_1 \to cR_1$, we have:

$$\Delta = \frac{1}{c}\begin{vmatrix} 0 & ac & -bc \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$$

Applying $R_1 \to R_1 - bR_2$, we have:

$$\Delta = \frac{1}{c}\begin{vmatrix} ab & ac & 0 \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$$

$$= \frac{a}{c}\begin{vmatrix} b & c & 0 \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$$

Here, the two rows R_1 and R_3 are identical.

$\therefore \Delta = 0.$

Question 7:

By using properties of determinants, show that:

$$\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 4a^2b^2c^2$$

Solution 7:

$$\Delta = \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix}$$

$$= abc\begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix} \qquad [\text{Taking out factors a, b, c from } R_1, R_2 \text{ and } R_3]$$

$$= = a^2b^2c^2\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \qquad [\text{Taking out factors a, b, c from } C_1, C_2 \text{ and } C_3]$$

Applying $R_2 \to R_2 + R_1$ and $R_3 \to R_3 + R_1$, we have:

$$\Delta = a^2b^2c^2\begin{vmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{vmatrix}$$

$$= a^2b^2c^2(-1)\begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix}$$

$$= -a^2b^2c^2(0-4) = 4a^2b^2c^2$$

Question 8:

By using properties of determinants, show that:

(i) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a+b)(b-c)(c-a)$

(ii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

Solution 8:

(i) Let $\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

Applying $R_1 \to R_1 - R_3$ and $R_2 \to R_2 - R_3$, we have:

$$\Delta = \begin{vmatrix} 0 & a-c & a^2-c^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= (c-a)(b-c)\begin{vmatrix} 0 & -1 & -a-c \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2$, we have:

$$\Delta = (b-c)(c-a)\begin{vmatrix} 0 & 0 & -a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$

$$= (a-b)(b-c)(c-a)\begin{vmatrix} 0 & 0 & -1 \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$

Expanding along C_1, we have:

$$\Delta=(a-b)(b-c)(c-a)\begin{vmatrix}0 & -1\\ 1 & b+c\end{vmatrix}=(a-b)(b-c)(c-a)$$

Hence, the given result is proved.

(ii) Let $\Delta=\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ a^3 & b^3 & c^3\end{vmatrix}$

Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$, we have:

$$\Delta=\begin{vmatrix}0 & 0 & 1\\ a-c & b-c & c\\ a^3-c^3 & b^3-c^3 & c^3\end{vmatrix}$$

$$=\begin{vmatrix}0 & 0 & 1\\ a-c & b-c & c\\ (a-c)\left(a^2+ac+c^2\right) & (b-c)\left(b^2+bc+^2\right) & c^3\end{vmatrix}$$

$$=(c-a)(b-c)\begin{vmatrix}0 & 0 & 1\\ -1 & 1 & c\\ -\left(a^2+ac+c^2\right) & \left(b^2+bc+c^2\right) & c^3\end{vmatrix}$$

Applying $C_1 \to C_1 + C_2$, we have:

$$\Delta=(c-a)(b-c)\begin{vmatrix}0 & 0 & 1\\ 0 & 1 & c\\ \left(b^2-a^2\right)+(bc-ac) & \left(b^2+bc+c^2\right) & c^3\end{vmatrix}$$

$$=(b-c)(c-a)(a-b)\begin{vmatrix}0 & 0 & 1\\ 0 & 0 & c\\ -(a+b+c) & \left(b^2+bc+c^2\right) & c^3\end{vmatrix}$$

$$=(a-b)(b-c)(c-a)(a+b+c)\begin{vmatrix}0 & 0 & 1\\ 0 & 1 & c\\ -1 & \left(b^2+bc+c^2\right) & c^3\end{vmatrix}$$

Expanding along C_1, we have:

$$\Delta=(a-b)(b-c)(c-a)(a+b+c)(-1)\begin{vmatrix}0 & 1\\1 & c\end{vmatrix}$$

$$=(a-b)(b-c)(c-a)(a+b+c)$$

Hence, the given result is proved.

Question 9:

By using properties of determinants, show that:

$$\begin{vmatrix}x & x^2 & yz\\y & y^2 & zx\\z & z^2 & xy\end{vmatrix}=(x-y)(y-z)(z-x)(xy+yz+zx)$$

Solution 9:

Let $\Delta=\begin{vmatrix}x & x^2 & yz\\y & y^2 & zx\\z & z^2 & xy\end{vmatrix}$

Applying $R_2 \to R_2 - R_1$, and $R_3 \to R_3 - R_1$, we have:

$$\Delta=\begin{vmatrix}x & x^2 & yz\\y-x & y^2-x^2 & zx-yz\\z-x & z^2-x^2 & xy-yz\end{vmatrix}$$

$$=\begin{vmatrix}x & x^2 & yz\\-(x-y) & -(x-y)(x+y) & z(x-y)\\(z-x) & (z-x)(z+x) & -y(z-x)\end{vmatrix}$$

$$=(x-y)(z-x)\begin{vmatrix}x & x^2 & yz\\-1 & -x-y & z\\1 & z-y & z-y\end{vmatrix}$$

Applying $R_3 \to R_3 + R_2$, we have:

$$\Delta=(x-y)(z-x)\begin{vmatrix}x & x^2 & yz\\-1 & -x-y & z\\1 & z-y & z-y\end{vmatrix}$$

$$=(x-y)(z-x)(z-y)\begin{vmatrix} x & x^2 & yz \\ -1 & -x-y & z \\ 0 & 1 & 1 \end{vmatrix}$$

Expanding along R_3, we have:

$$\Delta=\left[(x-y)(z-x)(z-y)\right]\left[(-1)\begin{vmatrix} x & yz \\ -1 & z \end{vmatrix}+1\begin{vmatrix} x & x^2 \\ -1 & -x-y \end{vmatrix}\right]$$

$$=(x-y)(z-x)(z-y)\left[(-xz-yz)+(-x^2-xy+x^2)\right]$$

$$=-(x-y)(z-x)(z-y)(xy+yz+zx)$$

$$=(x-y)(y-z)(z-x)(xy+yz+zx)$$

Hence, the given result is proved.

Question 10:

By using properties of determinants, show that:

(i) $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}=(5x+4)(4-x)^2$

(ii) $\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix}=k^2(3x+k)$

Solution 10:

(i) $\Delta=\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$

Applying $R_1 \to R_1 + R_2 + R_3$, we have:

$$\Delta=\begin{vmatrix} 5x+4 & 5x+4 & 5x+4 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

$$= (5x+4)\begin{vmatrix} 1 & 0 & 0 \\ 2x & x+4 & 0 \\ 2x & 0 & x+4 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$, we have

$$\Delta = (5x+4)\begin{vmatrix} 1 & 0 & 0 \\ 2x & -x+4 & 0 \\ 2x & 0 & -x+4 \end{vmatrix}$$

$$= (5x+4)(4-x)(4-x)\begin{vmatrix} 1 & 0 & 0 \\ 2x & 1 & 0 \\ 2x & 0 & 1 \end{vmatrix}$$

Expanding along C_3, we have:

$$\Delta = (5x+4)(4-x)^2\begin{vmatrix} 1 & 0 \\ 2x & 1 \end{vmatrix}$$

$$= (5x+4)(4-x)^2$$

Hence, the given result is proved.

(ii) $\Delta = \begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix}$

Applying $R_1 \to R_1 + R_2\ R_3$, we have:

$$\Delta = \begin{vmatrix} 3y+k & 3y+k & 3y+k \\ y & y+k & y \\ y & y & y+k \end{vmatrix}$$

$$= (3y+k)\begin{vmatrix} 1 & 1 & 1 \\ y & y+k & y \\ y & y & y+k \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we have:

$$\Delta = (3y+k)\begin{vmatrix} 1 & 0 & 0 \\ y & k & 0 \\ y & 0 & k \end{vmatrix}$$

$$= k^2(3x+k)\begin{vmatrix} 1 & 0 & 0 \\ y & 1 & 0 \\ y & 0 & 1 \end{vmatrix}$$

Expanding alone C_3, we have:

$$\Delta = k^2(3y+k)\begin{vmatrix}1 & 0\\ y & 1\end{vmatrix} = k^2(3y+k)$$

Hence, the given result is proved.

Question 11:

By using properties of determinants, show that:

(i) $\begin{vmatrix}a-b-c & 2a & 2a\\ 2b & b-c-a & 2b\\ 2c & 2c & c-a-b\end{vmatrix} = (a+b+c)^3$

(ii) $\begin{vmatrix}x+y+2z & x & y\\ z & y+z+2x & y\\ z & x & z+x+2y\end{vmatrix} = 2(x+y+z)^3$

Solution 11:

(i) $\begin{vmatrix}a-b-c & 2a & 2a\\ 2b & b-c-a & 2b\\ 2c & 2c & c-a-b\end{vmatrix} = (a+b+c)^3$

Applying $R_1 \to R_1 + R_2 + R_3$, we have:

$$\Delta = \begin{vmatrix}a+b+c & a+b+c & a+b+c\\ 2b & b-c-a & 2b\\ 2c & 2c & c-a-b\end{vmatrix}$$

$$=(a+b+c)\begin{vmatrix}1 & 1 & 1\\ 2b & b-c-a & 2b\\ 2c & 2c & c-a-b\end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we have:

$$\Delta = (a+b+c)\begin{vmatrix}1 & 0 & 0\\ 2b & -(a+b+c) & 0\\ 2c & 0 & -(a+b+c)\end{vmatrix}$$

$$=(a+b+c)^3\begin{vmatrix}1 & 0 & 0\\ 2b & -1 & 0\\ 2c & 0 & -1\end{vmatrix}$$

Expanding along C_3, we have:

$$\Delta = (a+b+c)^3(-1)(-1) = (a+b+c)^3$$

Hence, the given result is proved.

(ii) $\Delta = \begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$, we have:

$$\Delta = \begin{vmatrix} 2(x+y+z) & x & y \\ 2(x+y+z) & y+z+2x & y \\ 2(x+y+z) & x & z+x+2y \end{vmatrix}$$

$$= 2(x+y+z)\begin{vmatrix} 1 & x & y \\ 1 & y+z+2x & y \\ 1 & x & z+x+2y \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we have:

$$\Delta = 2(x+y+z)\begin{vmatrix} 1 & x & y \\ 0 & x+y+z & 0 \\ 0 & 0 & x+y+z \end{vmatrix}$$

$$= 2(x+y+z)\begin{vmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along R_3, we have:

$$\Delta = 2(x+y+z)^3(1)(1-0) = 2(x+y+z)^3$$

Hence, the given result is proved.

Question 12:

By using properties of determinants, show that:

$$\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1-x^3)^2$$

Solution 12:

$$\Delta = \begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$, we have:

$$\Delta = \begin{vmatrix} 1+x+x^2 & 1+x+x^2 & 1+x+x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we have:

$$\Delta = \left(1+x+x^2\right)\begin{vmatrix} 1 & 0 & 0 \\ x^2 & 1-x^2 & x-x^2 \\ x & x^2-x & 1-x \end{vmatrix}$$

$$= \left(1+x+x^2\right)\left(1-x\right)\left(1-x\right)\begin{vmatrix} 1 & 0 & 0 \\ x^2 & 1+x & x \\ x & -x & 1 \end{vmatrix}$$

$$= \left(1-x^3\right)\left(1-x\right)\begin{vmatrix} 1 & 0 & 0 \\ x^2 & 1+x & x \\ x & -x & 1 \end{vmatrix}$$

Expanding along R_1, we have:

$$\Delta = \left(1-x^3\right)\left(1-x\right)\left(1\right)\begin{vmatrix} 1+x & x \\ -x & 1 \end{vmatrix}$$

$$= \left(1-x^3\right)\left(1-x\right)\left(1+x+x^2\right)$$

$$= \left(1-x^3\right)\left(1-x^3\right)$$

$$= \left(1-x^3\right)^2$$

Hence, the given result is proved.

Question 13:

By using properties of determinants, show that:

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = \left(1+a^2+b^2\right)^3$$

Solution 13:

$$\Delta = \begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$$

Applying $R_1 \to R_1 + bR_3$ and $R_2 \to R_2 - aR_3$, we have:

$$\Delta=\begin{vmatrix}1+a^2+b^2 & 0 & -b\left(1+a^2+b^2\right)\\ 0 & 1+a^2+b^2 & a\left(1+a^2+b^2\right)\\ 2b & -2a & 1-a^2-b^2\end{vmatrix}$$

$$=\left(1+a^2+b^2\right)\begin{vmatrix}1 & 0 & -b\\ 0 & 1 & a\\ 2b & -2a & 1-a^2-b^2\end{vmatrix}$$

Expanding along R_1, we have:

$$\Delta=\left(1+a^2+b^2\right)^2\left[(1)\begin{vmatrix}1 & a\\ -2a & 1-a^2-b^2\end{vmatrix}-b\begin{vmatrix}0 & 1\\ 2b & -2a\end{vmatrix}\right]$$

$$=\left(1+a^2+b^2\right)^2\left[1-a^2-b^2+2a^2-b(-2b)\right]$$

$$=\left(1+a^2+b^2\right)^2\left(1+a^2+b^2\right)$$

$$=\left(1+a^2+b^2\right)^3$$

Question 14:

By using properties of determinants, show that:

$$\begin{vmatrix}a^2+1 & ab & ac\\ ab & b^2+1 & bc\\ ca & cb & c^2+1\end{vmatrix}=1+a^2+b^2+c^2$$

Solution 14:

$$\Delta=\begin{vmatrix}a^2+1 & ab & ac\\ ab & b^2+1 & bc\\ ca & cb & c^2+1\end{vmatrix}$$

Taking out common factors a, b and c from R_1, R_2 and R_3 respectively, we have:

$$\Delta=abc\begin{vmatrix}a+\frac{1}{a} & b & c\\ a & b+\frac{1}{b} & c\\ a & b & c+\frac{1}{c}\end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we have:

$$\Delta = abc\begin{vmatrix} a+\frac{1}{a} & b & c \\ -\frac{1}{a} & \frac{1}{b} & 0 \\ -\frac{1}{a} & 0 & \frac{1}{c} \end{vmatrix}$$

Applying $C_1 \to aC_1$, $C_2 \to bC_2$ and $C_3 \to cC_3$, we have:

$$\Delta = abc \text{ x } \frac{1}{\text{abc}}\begin{vmatrix} a^2+1 & b^2 & c^2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

Expanding along R_3, we have:

$$\Delta = -1\begin{vmatrix} b^2 & c^2 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} a^2+1 & b^2 \\ -1 & 1 \end{vmatrix}$$

$$= -1\left(-c^2\right) + \left(a^2+1+b^2\right) = 1+a^2+b^2+c^2$$

Hence, the given result is proved.

Question 15:

Choose the correct answer.

Let A be a square matrix of order $3 \text{ x } 3$, then $|kA|$ is equal to

A. $k|A|$

B. $k^2|A|$

C. $k^3|A|$

D. $3k|A|$

Solution 15:

Answer: C

A is a square matrix of order $3 \text{ x } 3$.

$$\text{Let } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

Then, $kA = \begin{bmatrix} ka_1 & kb_1 & kc_1 \\ ka_2 & kb_2 & kc_2 \\ ka_3 & kb_3 & kc_3 \end{bmatrix}$

$\therefore |kA| = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ ka_2 & kb_2 & kc_2 \\ ka_3 & kb_3 & kc_3 \end{vmatrix}$

$= k^3 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ **(Taking out common factors k from each row)**

$= k^3 |A|$

$\therefore |kA| = k^3 |A|$

Hence, the correct answer is C.

Question 16:

Which of the following is correct?

A. Determinant is a square matrix.
B. Determinant is a number associated to a matrix.
C. Determinant is a number associated to a square matrix.
D. None of these.

Solution 16:

Answer: C

We know that to every square matrix, $A = [aij]$ of order n. We can associate a number called the determinant of square matrix A, where $aij = (i, j)^{th}$ element of A.

Thus, the determinant is a number associated to a square matrix.

Hence, the correct answer is C.

Exercise 4.3

Question 1:
Find area of the triangle with vertices at the point given in each of the following:

(i) $(1,0)$, $(6,0)$,$(4,3)$

(ii) $(2,7)$, $(1,1)$, $(10,8)$

(iii) $(-2,-3)$, $(3,2)$, $(-1,-8)$

Solution 1:
(i) The area of the triangle with vertices $(1,0),(6,0),(4,3)$ is given by the relation,

$$\Delta=\frac{1}{2}\begin{vmatrix}1 & 0 & 1\\ 6 & 0 & 1\\ 4 & 3 & 1\end{vmatrix}$$

$$=\frac{1}{2}\left[1(0-3)-0(6-4)+1(18-0)\right]$$

$$=\frac{1}{2}[-3+18]=\frac{15}{2}\text{ square units}$$

(ii) The area of the triangle with vertices $(2,7),(1,1),(10,8)$ is given by the relation,

$$\Delta=\frac{1}{2}\begin{vmatrix}2 & 7 & 1\\ 1 & 1 & 1\\ 10 & 8 & 1\end{vmatrix}$$

$$=\frac{1}{2}\left[2(1-8)-7(1-10)+1(8-10)\right]$$

$$=\frac{1}{2}\left[2(-7)-7(-9)+1(-2)\right]$$

$$=\frac{1}{2}[-14+63-2]=\frac{1}{2}[-16+63]$$

$$=\frac{47}{2}\text{ square units}$$

(iii) The area of the triangle with vertices $(-2,-3)$, $(3,2)$, $(-1,-8)$ is given by the relation,

$$\Delta=\frac{1}{2}\begin{vmatrix}-2 & -3 & 1\\ 3 & 2 & 1\\ -1 & -8 & 1\end{vmatrix}$$

$$=\frac{1}{2}\left[-2(2+8)+3(3+1)+1(-24+2)\right]$$

$$=\frac{1}{2}\left[-2(10)+3(4)+1(-22)\right]$$

$$=\frac{1}{2}\left[-20+12-22\right]$$

$$=-\frac{30}{2}=-15$$

Hence, the area of the triangle is $|-15|=15$ square units

Question 2:

Show that points $A(a,b+c), B(b,c+a), C(c,a+b)$ are collinear

Solution 2:

Area of ΔABC is given by the relation,

$$\Delta=\frac{1}{2}\begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$$

$$=\frac{1}{2}\begin{vmatrix} a & b+c & 1 \\ b-a & a-b & 0 \\ c-a & a-c & 0 \end{vmatrix} \qquad \text{(Applying } R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1\text{)}$$

$$=\frac{1}{2}(a-b)(c-a)\begin{vmatrix} a & b+c & 1 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix}$$

$$=\frac{1}{2}(a-b)(c-a)\begin{vmatrix} a & b+c & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} \qquad \text{(Applying } R_3 \to R_3 + R_2\text{)}$$

$=0$ (All elements of R_3 are 0)

Thus, the area of the triangle formed by points A, B and C is zero.

Hence, the points A, B and C are collinear.

Question 3:

Find values of k if area of triangle is 4 square units and vertices are

(i) $(k,0),(4,0),(0,2)$

(ii) $(-2,0),(0,4),(0,k)$

Solution 3:

We know that the area of a triangle whose vertices are $(x_1,y_1),(x_2,y_2)$ and (x_3,y_3) is the absolute value of the determinant (Δ), where

$$\Delta=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

It is given that the area of triangle is 4 square units.

$\therefore \Delta = \pm 4$.

(i) The area of the triangle with vertices $(k,0),(4,0),(0,2)$ is given by the relation,

$$\Delta=\frac{1}{2}\begin{vmatrix} k & 0 & 1 \\ 4 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix}$$

$$=\frac{1}{2}\left[k(0-2)-0(4-0)+1(8-0)\right]$$

$$=\frac{1}{2}[-2k+8]=k+4$$

$\therefore -k + 4 = \pm 4$

When $-k + 4 = -4, k=8.$

When $-k + 4 = -4, k=0.$

Hence, $k=0,8$.

(ii) The area of the triangle with vertices $(-2,0),(0,4),(0,k)$ is given by the relation,

$$\Delta=\frac{1}{2}\begin{vmatrix} -2 & 0 & 1 \\ 0 & 4 & 1 \\ 0 & k & 1 \end{vmatrix}$$

$$=\frac{1}{2}\left[-2(4-k)\right]$$

$=k - 4$

$\therefore k - 4 = \pm 4$

When $k - 4 = -4, k=0.$

When $k - 4 = 4, k=8.$

Hence, $k=0,8$.

Question 4:

(i) Find equation of line joining $(1,2)$ and $(3,6)$ using determinates

(ii) Find equation of line joining $(3,1)$ and $(9,3)$ using determinants

Solution 4:

(i) Let $P(x,y)$ be any point on the line joining points $A(1,2)$ and $B(3,6)$. Then, the points A, B and P are collinear. Therefore, the area of triangle ABP will be zero.

$$\therefore \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 3 & 6 & 1 \\ x & y & 1 \end{vmatrix} = 0$$

$$\Rightarrow \frac{1}{2}\left[1(6-y)-2(3-x)+1(3y-6x)\right]=0$$

$$\Rightarrow 6-y-6+2x+3y-6x=0$$

$$\Rightarrow 2y-4x=0$$

$$\Rightarrow y=2x$$

Hence, the equation of the line joining the given points is $y=2x$.

(ii) Let $P(x,y)$ be any point on the line joining points $A(3,1)$ and $B(9,3)$. Then, the points A, B and P are collinear. Therefore, the area of the triangle ABP will be zero.

$$\therefore \frac{1}{2}\begin{vmatrix} 3 & 1 & 1 \\ 9 & 3 & 1 \\ x & y & 1 \end{vmatrix} = 0$$

$$\Rightarrow \frac{1}{2}\left[3(3-y)-1(9-x)+1(9y-3x)\right]=0$$

$$\Rightarrow 9-3y-9+x+9y-3x=0$$

$$\Rightarrow 6y-2x=0$$

$$\Rightarrow x-3y=0$$

Hence, the equation of the line joining the given points is $x-3y=0$.

Question 5:

If the area of triangle is 35 square units with vertices $(2,-6)$, $(5,4)$ and $(k,4)$. Then k is

A. 12

B. -2
C. -12, -2
D. 12, -2

Solution 5:

Answer: D

The area of the triangle with vertices $(2,-6)$, $(5,4)$ and $(k,4)$ is given by the relation,

$$\Delta = \frac{1}{2}\begin{vmatrix} 2 & -6 & 1 \\ 5 & 4 & 1 \\ k & 4 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[2(4-4)+6(5-k)+1(20-4k)\right]$$

$$= \frac{1}{2}[30-6k+20-4k]$$

$$= \frac{1}{2}[50-10k]$$

$$= 25-5k$$

It is given that the area of the triangle id ± 35.

Therefore, we have:

$\Rightarrow$ 25-5k= ± 35
$\Rightarrow$ 5(5-k)= ± 35
$\Rightarrow$ 5-k= ± 7

When $5-k=-7, k=5+7=12$.

When $5-k=-7, k=5-7=-2$.

Hence, $k=12, -2$.

The correct answer is D.

Exercise 4.4

Question 1:
Write Minors and Cofactors of the elements of following determinants:

(i) $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$

(ii) $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$

Solution 1:
(i) The given determinant is $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$.

Minor of element a_{ij} is M_{ij}.

$\therefore \mathrm{M}_{11} =$ minor of element $a_{11} = 3$

$\mathrm{M}_{12} =$ minor of element $a_{12} = 0$

$\mathrm{M}_{21} =$ minor of element $a_{21} = -4$

$\mathrm{M}_{22} =$ minor of element $a_{22} = 2$

Cofactor of a_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$.

$\therefore A_{11} = (-1)^{1+1} M_{11} = (-1)^2 (3) = 3$

$A_{12} = (-1)^{1+2} M_{12} = (-1)^3 (0) = 0$

$A_{21} = (-1)^{2+1} M_{21} = (-1)^3 (-4) = 4$

$A_{22} = (-1)^{2+2} M_{22} = (-1)^4 (2) = 2$

(ii) The given determinant is $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$.

Minor of element a_{ij} is M_{ij}.

$\therefore \mathrm{M}_{11} =$ minor of element $a_{11} = d$

$\mathrm{M}_{12} =$ minor of element $a_{12} = b$

$\mathrm{M}_{21} =$ minor of element $a_{21} = c$

$\mathrm{M}_{22} =$ minor of element $a_{22} = a$

Cofactor of a_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$.

$\therefore A_{11} = (-1)^{1+1} M_{11} = (-1)^2 (d) = d$

$$A_{12} = (-1)^{1+2} M_{12} = (-1)^3 (b) = -b$$
$$A_{21} = (-1)^{2+1} M_{21} = (-1)^3 (c) = -c$$
$$A_{22} = (-1)^{2+2} M_{22} = (-1)^4 (a) = a$$

Question 2:

Write Minors and Cofactors of the elements of following determinants:

(i)

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

(ii)

$$\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$$

Solution 2:

The given determinant is $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

By the definition of minors and cofactors, we have:

$$\text{M}_{11} = \text{minor of } a_{11} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$M_{12} = \text{minor of } a_{12} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$$

$$M_{13} = \text{minor of } a_{13} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0$$

$$M_{21} = \text{minor of } a_{21} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$$

$$M_{22} = \text{minor of } a_{22} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$M_{23} = \text{minor of } a_{23} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$$

$$M_{31} = \text{minor of } a_{31} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0$$

$$M_{32} = \text{minor of } a_{32} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$$

$$M_{33} = \text{minor of } a_{33} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$A_{11} = \text{cofactor of } a_{11} = (-1)^{1+1} M_{11} = 1$$

$$A_{12} = \text{cofactor of } a_{12} = (-1)^{1+2} M_{12} = 0$$

$$A_{13} = \text{cofactor of } a_{13} = (-1)^{1+3} M_{13} = 0$$

$$A_{21} = \text{cofactor of } a_{21} = (-1)^{2+1} M_{21} = 0$$

$$A_{22} = \text{cofactor of } a_{22} = (-1)^{2+2} M_{22} = 1$$

$$A_{23} = \text{cofactor of } a_{23} = (-1)^{2+3} M_{23} = 0$$

$A_{31} = \text{cofactor of } a_{31} = (-1)^{3+1} M_{31} = 0$

$A_{32} = \text{cofactor of } a_{32} = (-1)^{3+2} M_{32} = 0$

$A_{33} = \text{cofactor of } a_{33} = (-1)^{3+3} M_{33} = 1$

The given determinant is $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$

By definition of minors and cofactors, we have:

$M_{11} = \text{minor of } a_{11} = \begin{vmatrix} 5 & -1 \\ 1 & 2 \end{vmatrix} = 10 + 1 = 1$

$M_{12} = \text{minor of } a_{12} = \begin{vmatrix} 3 & -1 \\ 0 & 2 \end{vmatrix} = 6 - 0 = 6$

$M_{13} = \text{minor of } a_{13} = \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3 - 0 = 3$

$M_{21} = \text{minor of } a_{21} = \begin{vmatrix} 0 & 4 \\ 1 & 2 \end{vmatrix} = 0 - 4 = -4$

$M_{22} = \text{minor of } a_{22} = \begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix} = 2 - 0 = 2$

$M_{23} = \text{minor of } a_{23} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1$

$M_{31} = \text{minor of } a_{31} = \begin{vmatrix} 0 & 4 \\ 5 & -1 \end{vmatrix} = 0 - 20 = -20$

$M_{32} = \text{minor of } a_{32} = \begin{vmatrix} 1 & 4 \\ 3 & -1 \end{vmatrix} = -1 - 12 = -13$

$$M_{33} = \text{minor of } a_{33} = \begin{vmatrix} 1 & 0 \\ 3 & 5 \end{vmatrix} = 5 - 0 = 5$$

$$A_{11} = \text{cofactor of } a_{11} = (-1)^{1+1} M_{11} = 11$$

$$A_{12} = \text{cofactor of } a_{12} = (-1)^{1+2} M_{12} = -6$$

$$A_{13} = \text{cofactor of } a_{13} = (-1)^{1+3} M_{13} = 3$$

$$A_{21} = \text{cofactor of } a_{21} = (-1)^{2+1} M_{21} = 4$$

$$A_{22} = \text{cofactor of } a_{22} = (-1)^{2+2} M_{22} = 2$$

$$A_{23} = \text{cofactor of } a_{23} = (-1)^{2+3} M_{23} = -1$$

$$A_{31} = \text{cofactor of } a_{31} = (-1)^{3+1} M_{31} = -20$$

$$A_{32} = \text{cofactor of } a_{32} = (-1)^{3+2} M_{32} = 13$$

$$A_{33} = \text{cofactor of } a_{33} = (-1)^{3+3} M_{33} = 5$$

Question 3:

Using Cofactors of elements of second row, evaluate $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

Solution 3:

The given determinant is $\begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

We have:

$$M_{21} = \begin{vmatrix} 3 & 8 \\ 2 & 3 \end{vmatrix} = 9 - 16 = -7$$

$$\therefore A_{21} = \text{cofactor of } a_{21} = (-1)^{2+1} M_{21} = 7$$

$$M_{22} = \begin{vmatrix} 5 & 8 \\ 1 & 3 \end{vmatrix} = 15 - 8 = 7$$

$$\therefore A_{22} = \text{cofactor of } a_{22} = (-1)^{2+2} M_{22} = 7$$

$$M_{23} = \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = 10 - 3 = 7$$

$$\therefore A_{23} = \text{cofactor of } a_{23} = (-1)^{2+3} M_{23} = -7$$

We know that Δ is equal to the sum of the product of the elements of the second row with their corresponding cofactors.

$$\therefore \Delta = a_{21}A_{21} + a_{22}A_{22} + a_{33}A_{33}$$
$$= 2(7) + 0(7) + 1(-7) = 14 - 7 = 7$$

Question 4:

Using Cofactors of elements of third column, evaluate $\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$

Solution 4:

The given determinant is $\begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$

We have:

$$M_{13} = \begin{vmatrix} 1 & y \\ 1 & z \end{vmatrix} = z - y$$

$$M_{23} = \begin{vmatrix} 1 & x \\ 1 & z \end{vmatrix} = z - x$$

$$M_{33} = \begin{vmatrix} 1 & x \\ 1 & y \end{vmatrix} = y - x$$

$$\therefore A_{13} = \text{cofactor of } a_{13} = (-1)^{1+3} M_{13} = (z - y)$$

$A_{23} = \text{cofactor of } a_{23} = (-1)^{2+3} M_{23} = -(z-x) = (x-z)$

$A_{33} = \text{cofactor of } a_{33} = (-1)^{3+3} M_{33} = (y-x)$

We know that Δ is equal to the sum of the product of the elements of the second row with their corresponding cofactors.

$\Delta = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}$

$= yz(z-y) + zx(x-z) + xy(y-x)$

$= yz^2 - y^2z + x^2z - xz^2 + xy^2 - x^2y$

$= (x^2z - y^2z) + (yz^2 - xz^2) + (xy^2 - x^2y)$

$= z(x^2 - y^2) + z^2(y-x) + xy(y-x)$

$= z(x-y)(x+y) + z^2(y-x) + xy(y-x)$

$= (x-y)\left[zx + zy - z^2 - xy\right]$

$= (x-y)\left[z(x-z) + y(z-x)\right]$

$= (x-y)(z-x)[-z+y]$

$= (x-y)(y-z)(z-x)$

Hence, $\Delta = (x-y)(y-z)(z-x)$.

Question 5:

For the matrices A and B, verify that $(AB)' = B'A'$ where

(i) $A = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}, B = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 5 & 7 \end{bmatrix}$

Solution 5:

(i) $AB = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}\begin{bmatrix} -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 1 \\ 4 & -8 & -4 \\ -3 & 6 & 3 \end{bmatrix}$

$$\therefore (AB)' = \begin{bmatrix} -1 & 4 & -3 \\ 2 & -8 & 6 \\ 1 & -4 & 3 \end{bmatrix}$$

Now, $A' = \begin{bmatrix} 1 & -4 & 3 \end{bmatrix}, B' = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

$$\therefore B'A' = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -4 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 4 & -3 \\ 2 & -8 & 6 \\ 1 & -4 & 3 \end{bmatrix}$$

Hence, we have verified that $(AB)' = B'A'$.

(ii) $AB = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 5 & 7 \\ 2 & 10 & 14 \end{bmatrix}$

$$\therefore (AB)' = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 5 & 10 \\ 0 & 7 & 14 \end{bmatrix}$$

Now, $A' = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}, B' = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}$

$$\therefore B'A' = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 5 & 10 \\ 0 & 7 & 14 \end{bmatrix}$$

Hence, we have verified that $(AB)' = B'A'$.

Exercise 4.5

Question 1:

Find adjoint of each of the matrices. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Solution 1:

Let $A=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

We have,

$A_{11}=4, A_{12}=-3, A_{13}=-2, A_{22}=1$

$\therefore\ adjA=\begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix}=\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

Question 2:

Find adjoint of each of the matrices $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$

Solution 2:

Let $A=\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$.

We have,

$A_{11}=\begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix}=3-0=3$

$A_{12}=\begin{vmatrix} 2 & 5 \\ -2 & 1 \end{vmatrix}=-(2+10)=-12$

$A_{13}=\begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix}=0+6=6$

$A_{21}=\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix}=-(-1-0)=1$

$A_{22}=\begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix}=1+4=5$

$$A_{23} = -\begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = -(0-2) = 2$$

$$A_{31} = \begin{vmatrix} -1 & 2 \\ 2 & 5 \end{vmatrix} = -5-6 = -11$$

$$A_{32} = -\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = -(5-4) = -1$$

$$A_{33} = \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 3+2 = 5$$

Hence, $adjA = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{bmatrix}$

Question 3:

Verify $A(adj\ A) = (adj\ A)A = |A|\, I$

$$\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$$

Solution 3:

$$A = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$$

We have,

$$|A| = -12-(-12) = -12+12 = 0$$

$$\therefore |A| I = 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Now,

$$\mathrm{A}_{11} = -6, \mathrm{A}_{12} = 4, \mathrm{A}_{21} = -3, \mathrm{A}_{22} = 2,$$

$$\therefore \text{adj A} = \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}$$

Now,

$$A(adj\ A)=\begin{bmatrix}2 & 3\\ -4 & -6\end{bmatrix}\begin{bmatrix}-6 & -3\\ 4 & 2\end{bmatrix}$$

$$=\begin{bmatrix}-12+12 & -6+6\\ 24-24 & 12-12\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

Also, $(adj\ A)A=\begin{bmatrix}-6 & -3\\ 4 & 2\end{bmatrix}\begin{bmatrix}2 & 3\\ -4 & -6\end{bmatrix}$

$$=\begin{bmatrix}-12+12 & -18+18\\ 8-8 & 12-12\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

Hence, $A(adj\ A)=(adj\ A)A=|A|I$.

Question 4:

Verify $A(adj\ A)=(adj\ A)A=|A|I$.

$$\begin{bmatrix}1 & -1 & 2\\ 3 & 0 & -2\\ 1 & 0 & 3\end{bmatrix}$$

Solution 4:

$$A=\begin{bmatrix}1 & -1 & 2\\ 3 & 0 & -2\\ 1 & 0 & 3\end{bmatrix}$$

$$|A|=1(0-0)+1(9+2)+2(0-0)=11$$

$$\therefore |A|I=11\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}11 & 0 & 0\\ 0 & 11 & 0\\ 0 & 0 & 11\end{bmatrix}$$

Now,

$$A_{11}=0, A_{12}=-(9+2)=-11, A_{13}=0$$

$$A_{21}=-(-3-0)=3, A_{22}=3-2=1, A_{22}=-(0+1)=-1$$

$$A_{31}=2-0=2, A_{32}=-(-2-6)=8, A_{33}=0+3=3$$

$$\therefore adj\ A=\begin{bmatrix}0 & 3 & 2\\ -11 & 1 & 8\\ 0 & -1 & 3\end{bmatrix}$$

Now,

$$A(adj\ A)=\begin{bmatrix}1 & -1 & 2\\3 & 0 & -2\\1 & 0 & 3\end{bmatrix}\begin{bmatrix}0 & 3 & 2\\-11 & 1 & 8\\0 & -1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}0+11+0 & 3-1-2 & 2-8+6\\0+0+0 & 9+0+2 & 6+0-6\\0+0+0 & 3+0-3 & 2+0+9\end{bmatrix}$$

$$=\begin{bmatrix}11 & 0 & 0\\0 & 11 & 0\\0 & 0 & 11\end{bmatrix}$$

Also,

$$(adj\ A).A=\begin{bmatrix}0 & 3 & 2\\-11 & 1 & 8\\0 & -1 & 3\end{bmatrix}\begin{bmatrix}1 & -1 & 2\\3 & 0 & -2\\1 & 0 & 3\end{bmatrix}$$

$$=\begin{bmatrix}0+9+2 & 0+0+0 & 0-6+6\\-11+3+8 & 11+0+0 & -22-2+24\\0-3+3 & 0+0+0 & 0+2+9\end{bmatrix}$$

$$=\begin{bmatrix}11 & 0 & 0\\0 & 11 & 0\\0 & 0 & 11\end{bmatrix}$$

Hence, $A(adj\ A)=(adj\ A)A=A=|A|I$.

Question 6:

Find the inverse of each of the matrices (if it exists). $\begin{bmatrix}-1 & 5\\-3 & 2\end{bmatrix}$

Solution 6:

Let $A=\begin{bmatrix}-1 & 5\\-3 & 2\end{bmatrix}$

We have,

$|A|=-2+15=13$

Now,

$A_{11}=2, A_{12}=3, A_{21}=-5, A_{22}=-1$

$\therefore adjA=\begin{bmatrix}2 & -5\\ 3 & -1\end{bmatrix}$

$\therefore A^{-1}=\frac{1}{|A|}adjA=\frac{1}{13}\begin{bmatrix}2 & -5\\ 3 & -1\end{bmatrix}$

Question 7:

Find the inverse of each of the matrices (if it exists). $\begin{bmatrix}1 & 2 & 3\\ 0 & 2 & 4\\ 0 & 0 & 5\end{bmatrix}$

Solution 7:

Let $A=\begin{bmatrix}1 & 2 & 3\\ 0 & 2 & 4\\ 0 & 0 & 5\end{bmatrix}$

We have,

$|A|=1(10-0)-2(0-0)+3(0-0)=10$

Now,

$A_{11}=10-0, A_{12}=-(0-0)=0, A_{13}=0-0=0$

$A_{21}=-(10-0)=-10, A_{22}=5-0=5, A_{23}=-(0-0)=0$

$A_{31}=8-6=2, A_{32}=-(4-0)=-4, A_{33}=2-0=2$

$\therefore adjA=\begin{bmatrix}10 & -10 & 2\\ 0 & 5 & -4\\ 0 & 0 & 2\end{bmatrix}$

$\therefore A^{-1}=\frac{1}{|A|}adjA=\frac{1}{10}\begin{bmatrix}10 & -10 & 2\\ 0 & 5 & -4\\ 0 & 0 & 2\end{bmatrix}$

Question 8:

Find the inverse of each of the matrices (if it exists). $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$

Solution 8:

Let A= $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$

We have,

$$|A| = 1(-3-0) - 0 + 0 = -3$$

Now,

$$A_{11} = -3-0 = -3, A_{12} = -(-3-0) = 3, A_{13} = 6-15 = -9$$

$$A_{22} = -(0-0) = 0, A_{22} = -1-0 = -1, A_{22} = -(2-0) = -2$$

$$A_{31} = 0-0 = 0, A_{32} = -(0-0) = 0, A_{33} = 3-0 = 3$$

$$\therefore adjA = \begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} adjA = \frac{1}{3} \begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$$

Question 9:

Find the inverse of each of the matrices (if it exists). $\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$

Solution 9:

Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$

We have,

$$|A| = 2(-1-0) - 1(4-0) + 3(8-7)$$

$$= 2(-1) - 1(4) + 3(1)$$
$$= -2 - 4 + 3$$
$$= -3$$

Now,

$A_{11} = -1 - 0 = -1, A_{12} = -(4-0) = -4, A_{13} = 8 - 7 = 1$

$A_{22} = -(1-6) = 5, A_{22} = 2 + 21 = 23, A_{23} = -(4+7) = -11$

$A_{31} = 0 + 3 = 3, A_{22} = -(0-12) = 12, A_{33} = -2 - 4 = -6$

$$\therefore adjA = \begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} adjA = -\frac{1}{3}\begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$$

Question 10:

Find the inverse of each of the matrices (if it exists). $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$

Solution 10:

Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$

By expanding along C1, we have:

$|A| = 1(8-6) - 0 + 3(3-4) = 2 - 3 = -1$

Now,

$A_{11} = 8 - 6 = 2, A_{12} = -(0+9) = -9, A_{13} = 0 - 6 = -6$

$A_{21} = -(-4+4) = 0, A_{22} = 4 - 6 = -2, A_{23} = -(-2+3) = -1$

$A_{31} = 3 - 4 = -1, A_{32} = -(-3-0) = 3, A_{33} = 2 - 0 = 2$

$$\therefore adjA = \begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} adjA = -1\begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$$

Question 11:

Find the inverse of each of the matrices (if it exists). $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos a & \sin a \\ 0 & \sin a & -\cos a \end{bmatrix}$

Solution 11:

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos a & \sin a \\ 0 & \sin a & -\cos a \end{bmatrix}$ a

We have,

$$|A| = 1\left(-\cos^2 a - \sin^2 a\right) = -\left(\cos^2 a + \sin^2 a\right) = -1$$

Now,

$$A_{11} = -\cos^2 a - \sin^2 a = -1, A_{12} = 0, A_{13} = 0$$
$$A_{21} = 0, A_{22} = -\cos a, A_{23} = -\sin a$$
$$A_{31} = 0, A_{32} = -\sin a, A_{33} = \cos a$$

$$\therefore adjA = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos a & -\sin a \\ 0 & -\sin a & \cos a \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} adjA = -1 \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos a & -\sin a \\ 0 & -\sin a & \cos a \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos a & \sin a \\ 0 & \sin a & -\cos a \end{bmatrix}$$

Question 12:

Let $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$. Verify that $(AB)^{-1} = B^{-1}A^{-1}$

Solution 12:

Let $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$

We have,

$|A| = 15 - 14 = 1$

Now,

$A_{11} = 5, A_{12} = -2, A_{21} = -7, A_{22} = 3$

$$\therefore adjA = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \cdot \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$$

Now, let $B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$

We have,

$|B| = 54 - 56 = -2$

$\therefore adjB = \begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}$

$\therefore B^{-1} = \frac{1}{|B|} \bullet \qquad \frac{1}{2}\begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix} = \begin{bmatrix} -\frac{9}{2} & 4 \\ \frac{7}{2} & -3 \end{bmatrix}$

Now,

$B^{-1}A^{-1} = \begin{bmatrix} -\frac{9}{2} & 4 \\ \frac{7}{2} & -3 \end{bmatrix}\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$

$= \begin{bmatrix} -\frac{45}{2} - 8 & \frac{63}{2} + 12 \\ \frac{35}{2} + 6 & -\frac{49}{2} - 9 \end{bmatrix} = \begin{bmatrix} -\frac{61}{2} & \frac{87}{2} \\ \frac{47}{2} & -\frac{67}{2} \end{bmatrix} \qquad \ldots\ldots(1)$

Then,

$AB = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$

$= \begin{bmatrix} 18+49 & 24+63 \\ 12+35 & 16+45 \end{bmatrix}$

$= \begin{bmatrix} 67 & 87 \\ 47 & 61 \end{bmatrix}$

Therefore, we have $|AB|$ = 67x61-87x47=4087-4089=-2.

Also,

$\therefore adj\left(AB\right) = \begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix}$

$\therefore \left(AB\right)^{-1} = \frac{1}{|AB|} adj\left(AB\right) = -\frac{1}{2}\begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix}$

$$= \begin{bmatrix} -\frac{61}{2} & \frac{87}{2} \\ \frac{47}{2} & -\frac{67}{2} \end{bmatrix} \qquad \ldots\ldots.(2)$$

From (1) and (2), we have:

$(AB)^{-1} = B^{-1}A^{-1}$

Hence, the given result is proved.

Question 13:

If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$,show that $A^2 - 5A + 71 = O$. Hence find A^{-1}

Solution 13:

$A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$A^2 = A.A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$

$\therefore A^2 - 5A + 71$

$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

$= \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Hence $A^2 - 5A + 71 = O.$

$\therefore A - A - 5A = -71$

$\Rightarrow A\bullet \qquad 5AA^{-1} = -71A^{-1}$ $\qquad \left[\text{Post-multiplying by } A^{-1} as \left|A\right| \neq 0\right]$

$\Rightarrow A\left(AA^{-1}\right)-5I=-7A^{-1}$

$\Rightarrow AI-5I=-7A^{-1}$

$\Rightarrow A^{-1}=-\frac{1}{7}(A-5I)$

$\Rightarrow A^{-1}=\frac{1}{7}(5I-A)$

$=\frac{1}{7}\left(\begin{bmatrix}5 & 0\\ 0 & 5\end{bmatrix}-\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}\right)=\frac{1}{7}\begin{bmatrix}2 & -1\\ 1 & 3\end{bmatrix}$

$\therefore A^{-1}=\frac{1}{7}\begin{bmatrix}2 & -1\\ 1 & 3\end{bmatrix}$

Question 14:

For the matrix $A=\begin{bmatrix}3 & 2\\ 1 & 1\end{bmatrix}$.find the number a and b such that $A^2+aA+bI=0$.

Solution 14:

$A=\begin{bmatrix}3 & 2\\ 1 & 1\end{bmatrix}$

$\therefore A^2=\begin{bmatrix}3 & 2\\ 1 & 1\end{bmatrix}\begin{bmatrix}3 & 2\\ 1 & 1\end{bmatrix}=\begin{bmatrix}9+2 & 6+2\\ 3+1 & 2+1\end{bmatrix}=\begin{bmatrix}11 & 8\\ 4 & 3\end{bmatrix}$

Now,

$A^2+aA+bI=O$

$\Rightarrow (AA)A^{-1}+aAA^{-1}+bIA^{-1}=O$ $\left[\text{Post-multiplying by } A^{-1} as\ |A|\neq 0\right]$

$\Rightarrow A\left(AA^{-1}\right)+aI+b\left(IA^{-1}\right)=O$

$\Rightarrow AI+aI+bA^{-1}=O$

$\Rightarrow A+aI=-bA^{-1}$

$\Rightarrow A^{-1}=\frac{1}{b}(A+aI)$

Now,

$A^{-1}=\frac{1}{|A|}adjA=\frac{1}{1}\begin{bmatrix}1 & -2\\ -1 & 3\end{bmatrix}=\begin{bmatrix}1 & -2\\ -1 & 3\end{bmatrix}$

We have:

$$\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = -\frac{1}{b}\left(\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\right) = -\frac{1}{b}\begin{bmatrix} 3+a & 2 \\ 1 & 1+a \end{bmatrix} = \begin{bmatrix} \frac{-3-a}{b} & -\frac{2}{b} \\ -\frac{1}{b} & \frac{-1-a}{b} \end{bmatrix}$$

Comparing the corresponding elements of the two matrices, we have:

$$-\frac{1}{b} = -1 \Rightarrow b = 1$$

$$\frac{-3-a}{b} = 1 \Rightarrow -3-a \Rightarrow a = -4$$

Hence, -4 and 1 are the required values of a and b respectively.

Question 15:

For the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$ show that $A^3 - 6A^2 + 5A + 11\,I = 0$. Hence, A^{-1}.

Solution 15:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+2 & 1+2-1 & 1-3+3 \\ 1+2-6 & 1+4+3 & 1-6-9 \\ 2-1+6 & 2-2-3 & 2+3+9 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+2+2 & 4+4-1 & 4-6+3 \\ -3+8-28 & -3+16+14 & -3-24-42 \\ 7-3+28 & 7-6-14 & 7+9+42 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix}$$

$\therefore A^3 - 6A^2 + 5A + 11I$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - 6\begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} + 5\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix} + \begin{bmatrix} 5 & 5 & 5 \\ 5 & 10 & -15 \\ 2 & -5 & 15 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix} - \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Thus, $A^2 - 6A^2 + 5A + 11I = O.$

Now,

$A^3 - 6A^2 + 5A + 11I = O.$

$\Rightarrow (AAA)A^{-1} - 6(AA)A^{-1} + 5AA^{-1} + 11IA^{-1} = 0$ $\quad \left[\text{Post-multiplying by } A^{-1} as\ |A| \neq 0\right]$

$\Rightarrow AA\left(AA^{-1}\right) - 6A\left(AA^{-1}\right) + 5\left(AA^{-1}\right) = -11\left(IA^{-1}\right)$

$\Rightarrow A^2 - 6A + 5I = -11A^{-1}$

$\Rightarrow A^{-1} = -\dfrac{1}{11}\left(A^2 - 6A + 5I\right)$ …..(1)

Now,

$A^2 - 6A + 5I$

$$=\begin{bmatrix}4 & 2 & 1\\ -3 & 8 & -14\\ 7 & -3 & 14\end{bmatrix}-6\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}+5\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}4 & 2 & 1\\ -3 & 8 & -14\\ 7 & -3 & 14\end{bmatrix}-\begin{bmatrix}6 & 6 & 6\\ 6 & 12 & -18\\ 12 & -6 & 18\end{bmatrix}+\begin{bmatrix}5 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 5\end{bmatrix}$$

$$=\begin{bmatrix}9 & 2 & 1\\ -3 & 13 & -14\\ 7 & -3 & 19\end{bmatrix}-\begin{bmatrix}6 & 6 & 6\\ 6 & 12 & -18\\ 12 & -6 & 18\end{bmatrix}$$

$$=\begin{bmatrix}3 & -4 & -5\\ -9 & 1 & 4\\ -5 & 3 & 1\end{bmatrix}$$

From equation (1), we have:

$$A^{-1}=-\frac{1}{11}\begin{bmatrix}3 & -4 & -5\\ -9 & 1 & 4\\ -5 & 3 & 1\end{bmatrix}=\frac{1}{11}\begin{bmatrix}-3 & 4 & 5\\ 9 & -1 & -4\\ 5 & -3 & -1\end{bmatrix}$$

Question 16:

If $A=\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$ verify that $A^3-6A^2+9A-4I=0$ and hence find A^{-1}

Solution 16:

$$A=\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$$

$$A^2=\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$$

$$=\begin{bmatrix}4+1+1 & -2-2-1 & 2+1+2\\ -2-2-1 & 1+4+1 & -1-2-2\\ 2+1+2 & -1-2-2 & 1+1+4\end{bmatrix}$$

$$=\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix}$$

$$A^3=A^2A=\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix}\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$$

$$=\begin{bmatrix}12+5+5 & -6-10-5 & 6+5+10\\ -10-6-5 & 5+12+5 & -5-6-10\\ 10+5+6 & -5-10-6 & 5+5+12\end{bmatrix}$$

$$=\begin{bmatrix}22 & -21 & 21\\ -21 & 22 & -21\\ 21 & -21 & 22\end{bmatrix}$$

Now,

$$A^3-6A^2+9A-4I$$

$$=\begin{bmatrix}22 & -21 & 21\\ -21 & 22 & -21\\ 21 & -21 & 22\end{bmatrix}-6\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix}+9\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}-4\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}22 & -21 & 21\\ -21 & 22 & -21\\ 21 & -21 & 22\end{bmatrix}-\begin{bmatrix}36 & -30 & 30\\ -30 & 36 & -30\\ 30 & -30 & 36\end{bmatrix}+\begin{bmatrix}18 & -9 & 9\\ -9 & 18 & -9\\ 9 & -9 & 18\end{bmatrix}-\begin{bmatrix}4 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 4\end{bmatrix}$$

$$=\begin{bmatrix}40 & -30 & 30\\ -30 & 40 & -30\\ 30 & -30 & 40\end{bmatrix}-\begin{bmatrix}40 & -30 & 30\\ -30 & 40 & -30\\ 30 & -30 & 40\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$\therefore A^3-6A^2+9A-4I=0$$

Now,

$$A^3-6A^2+9A-4I=0$$

$$\Rightarrow (AAA)A^{-1}-6(AA^{-1})+9AA^{-1}-4IA^{-1}=0 \qquad \left[\text{Post-multiplying by } A^{-1} as\ |A|\neq 0\right]$$

$$\Rightarrow AA(AA^{-1})-6A(AA^{-1})+9(AA^{-1})=4(IA^{-1})$$

$$\Rightarrow AAI-6AI+9I=4A^{-1}$$

$$\Rightarrow A^2-6A+9I=4A^{-1}$$

$$\Rightarrow A^{-1}=\frac{1}{4}\left(A^2-6A+9I\right) \qquad \text{.....(1)}$$

$A^2 - 6A + 9I$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - \begin{bmatrix} 12 & -6 & 6 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

From equation (1), we have:

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Question 17:

Let A be nonsingular square matrix of order $3 \text{ x } 3$. Then $|adjA|$ is equal to

A. $|A|$

B. $|A|^2$

C. $|A|^3$

D. $3|A|$

Solution 17:

We know that,

$$(adjA) = A = |A|I = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}$$

$$\Rightarrow |(adjA)A| = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}$$

$$\Rightarrow |adjA||A| = |A|^3 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = |A|^3 (I)$$

$\therefore |adjA| = |A|^2$

Hence, the correct answer is B.

Question 18:

If A is an invertible matrix of order 2, then det $\left(A^{-1}\right)$ is equal to

A. $\det(A)$

B. $\dfrac{1}{\det(A)}$

C. 1

D. 0

Solution 18:

Since A is an invertible matrix, A^{-1} exists and $A^{-1} = \dfrac{1}{|A|} adjA.$

As matrix A is of order 2, let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$

Then, $|A| = ad - bc$ and $adjA = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$

Now,

$$A^{-1} = \frac{1}{|A|} adjA = \begin{bmatrix} \dfrac{d}{|A|} & \dfrac{-b}{|A|} \\ \dfrac{-c}{|A|} & \dfrac{a}{|A|} \end{bmatrix}$$

$$\therefore \left|A^{-1}\right| = \begin{bmatrix} \frac{d}{|A|} & \frac{-b}{|A|} \\ \frac{-c}{|A|} & \frac{a}{|A|} \end{bmatrix}$$

$$= \frac{1}{\left|A^2\right|}\begin{vmatrix} d & -b \\ -c & a \end{vmatrix}$$

$$= \frac{1}{\left|A^2\right|}(ad - bc)$$

$$= \frac{1}{\left|A^2\right|} \cdot |A| = \frac{1}{|A|}$$

$$\therefore \det\left(A^{-1}\right) = \frac{1}{\det(A)}$$

Hence, the correct answer is B.

Exercise 4.6

Question 1:

Examine the consistency of the system of equations.

$x + 2y = 2$

$2x + 3y = 3$

Solution 1:

The given system of equations is:

$x + 2y = 2$

$2x + 3y = 3$

The given system of equations can be written in the form of $AX = B$, where

$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, X = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Now,

$|A| = 1(3) - 2(2) = 3 - 4 = -1 \neq 0$

$\therefore A$ is non-singular.

Therefore, A^{-1} exists.

Hence, the given system of equations is consistent.

Question 2:

Examine the consistency of the system of equations.

$2x - y = 5$

$x + y = 4$

Solution 2:

The given system of equations is:

$2x - y = 5$

$x + y = 4$

The given system of equation can be written in the form of $AX = B$,where

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ z \end{bmatrix} \text{and } B = \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

$$|A| = 2(1) - (-1)(1) = 2 + 1 = 3 \neq 0$$

$\therefore A$ is non-singular.

Therefore, A^{-1} exists.

Hence, the given system of equations is consistent.

Question 3:

Examine the consistency of the system of equations.

$x + 3y = 5$

$2x + 6y = 8$

Solution 3:

The given system of equations is:

$x + 3y = 5$

$2x + 6y = 8$

The given system of equation can be written in the form of $AX = B$, where

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{and } B = \begin{bmatrix} 5 \\ 8 \end{bmatrix}.$$

Now,

$|A| = 1(6) - 3(2) = 6 - 6 = 0$

$\therefore A$ is a singular matrix.

$$(adjA) = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$$

$$(adjA)B = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 30-24 \\ -10+8 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix} \neq 0$$

Thus, the solution of the given system of equations does not exists, Hence, the given system of equations is inconsistent.

Question 4:

Examine the consistency of the system of equations.

$x + y + z = 1$

$2x + 3y + 2z = 2$

$ax + ay + 2az = 4$

Solution 4:

The given system of equations is:

$x + y + z = 1$

$2x + 3y + 2z = 2$

$ax + ay + 2az = 4$

The system of equation can be written in the form of $AX = B$,where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ a & a & 2a \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}.$$

Now,

$$|A| = 1(6a - 2a) - 1(4a - 2a) + 1(2a - 3a)$$
$$= 4a - 2a - a = 4a - 3a = a \neq 0$$

$\therefore A$ is a non-singular matrix.

Therefore, A^{-1} exists.

Hence, the given system of equation is consistent.

Question 5:
Examine the consistency of the system of equations.

$3x - y - 2z = 2$

$2y - z = -1$

$3x - 5y = 3$

Solution 5:
The given system of equation is:

$3x - y - 2z = 2$

$2y - z = -1$

$3x - 5y = 3$

This system of equations can be written in the form of $AX = B$, where

$$A = \begin{bmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

Now,

$$|A| = 3(-5) - 0 + 3(1+4) = -15 + 15 = 0$$

$\therefore$ A is a singular matrix.

Now,

$$(adjA) = \begin{bmatrix} -5 & 10 & 5 \\ -3 & 6 & 3 \\ -6 & 12 & 6 \end{bmatrix}$$

$$\therefore (adjA)B = \begin{bmatrix} -5 & 10 & 5 \\ -3 & 6 & 3 \\ -6 & 12 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -10 - 10 + 15 \\ -6 - 6 + 9 \\ -12 - 12 + 18 \end{bmatrix} = \begin{bmatrix} -5 \\ -3 \\ -6 \end{bmatrix} \neq 0$$

Thus, the solution of the given system of equation does not exist. Hence, the system of equations is inconsistent.

Question 6:

Examine the consistency of the system of equations.

$5x - y + 4z = 5$

$2x + 3y + 5z = 2$

$5x - 2y + 6z = -1$

Solution 6:

The given system of equation is:

$5x - y + 4z = 5$

$2x + 3y + 5z = 2$

$5x - 2y + 6z = -1$

The system of equation can be written in the form of $AX = B$,where

$$A = \begin{bmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 3 & -2 & 6 \end{bmatrix}, X \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}.$$

Now,

$|A| = 5(18+10) + 1(12-25) + 4(-4-15)$

$= 5(28) + 1(-13) + 4(-19)$

$= 140 - 13 - 76$

$= 51 \neq 0$

$\therefore A$ is non-singular.

Therefore, A^{-1} exists.

Hence, the given system of equations is consistent.

Question 7:

Solve system of linear equations, using matrix method.

$5x + 2y = 4$

$7x + 3y = 5$

Solution 7:

The given system of equation can be written in the form of $AX = B$, where

$A=\begin{bmatrix}5 & 2\\7 & 3\end{bmatrix}, X=\begin{bmatrix}x\\y\end{bmatrix}$ and $B=\begin{bmatrix}4\\5\end{bmatrix}$.

Now $|A|$=15-14=1 $\neq 0$

Thus, A is non-singular. Therefore, its inverse exists.

Now,

$$A^{-1}=\frac{1}{|\mathrm{A}|}(adjA)$$

$$\therefore A^{-1}=\begin{bmatrix}3 & -2\\-7 & 5\end{bmatrix}$$

$$\therefore X=A^{-1}B=\begin{bmatrix}3 & -2\\-7 & 5\end{bmatrix}\begin{bmatrix}4\\5\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}12-10\\-28+25\end{bmatrix}=\begin{bmatrix}2\\-3\end{bmatrix}$$

Hence, $x=2$ and $y=-3$.

Question 8:

Solve system of linear equations, using matrix method.

$2x-y=-2$

$3x+4y=3$

Solution 8:

The given system of equation can be written in the form of $AX=B$,where

$A=\begin{bmatrix}2 & -1\\3 & 4\end{bmatrix}, X=\begin{bmatrix}x\\y\end{bmatrix}$ and $B=\begin{bmatrix}-2\\3\end{bmatrix}$.

Now,

Now $|A|$=8+3=11 $\neq 0$

Thus, A is non-singular. Therefore, its inverse exists.

Now,

$$A^{-1}=\frac{1}{|A|}(adjA)=\frac{1}{11}\begin{bmatrix}4 & 1\\ -3 & 2\end{bmatrix}$$

$$\therefore X=A^{-1}B=\frac{1}{11}\begin{bmatrix}4 & 1\\ -3 & 2\end{bmatrix}\begin{bmatrix}-2\\ 3\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x\\ y\end{bmatrix}=\frac{1}{11}\begin{bmatrix}-8+3\\ 6+6\end{bmatrix}=\frac{1}{11}\begin{bmatrix}-5\\ 12\end{bmatrix}=\begin{bmatrix}-\frac{5}{11}\\ \frac{12}{11}\end{bmatrix}$$

Hence, $x=\frac{-5}{11}$ and $y=\frac{12}{11}$.

Question 9:

Solve system of linear equations, using matrix method.

$4x-3y=3$

$3x-5y=7$

Solution 9:

The given system of equation can be written in the form of $AX = B$,where

$A=\begin{bmatrix}4 & -3\\ 3 & -5\end{bmatrix}, X=\begin{bmatrix}x\\ y\end{bmatrix}$ and $B=\begin{bmatrix}3\\ 7\end{bmatrix}$

Now,

$|A|=-20+9=-11\neq 0$

Thus, A is non-singular. Therefore, its inverse exists.

Now,

$$A^{-1}=\frac{1}{|A|}(adjA)=-\frac{1}{11}\begin{bmatrix}-5 & 3\\ -3 & 4\end{bmatrix}=\frac{1}{11}\begin{bmatrix}5 & -3\\ 3 & -4\end{bmatrix}$$

$$\therefore X = A^{-1}B=\frac{1}{11}\begin{bmatrix}5 & -3\\ 3 & -4\end{bmatrix}\begin{bmatrix}3\\ 7\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 5 & -3 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix} 15-21 \\ 9-28 \end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix} -6 \\ -19 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{6}{11} \\ -\frac{19}{11} \end{bmatrix}$$

Hence, $x = \frac{-6}{11}$ and $y = \frac{-19}{11}$

Question 10:

Solve system of linear equations, using matrix method.

$5x + 2y = 3$
$3x + 2y = 5$

Solution 10:

The given system of equation can be written in the form of $AX = B$,where

$A = \begin{bmatrix} 5 & 2 \\ 3 & 2 \end{bmatrix}, X \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

Now,

$|A| = 10 - 6 = 4 \neq 0$

Thus A is non-singular, Therefore, its inverse exists.

Question 11:

Solve system of linear equations, using matrix method.

$$2x+y+z=1$$

$$x-2y-z=\frac{3}{2}$$

$$3y-5z=9$$

Solution 11:

The given system of equation can be written in the form of $AX = B$,where

$$A=\begin{bmatrix}2 & 1 & 1\\ 1 & -2 & -1\\ 0 & 3 & -5\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix} \text{ and } B=\begin{bmatrix}1\\ \frac{3}{2}\\ 9\end{bmatrix}.$$

Now,

$$|A|=2(10+3)-1(-5-3)+0=2(13)-1(-8)=26+8=34\neq 0$$

Thus A is non-singular. Therefore, its inverse exists.

Now,

$$A_{11}=13, A_{12}=5, A_{13}=3$$

$$A_{21}=8, A_{22}=-10, A_{23}=-6$$

$$A_{31}=1, A_{32}=3, A_{33}=-5$$

$$\therefore A^{-1}=\frac{1}{|A|}(adjA)=\frac{1}{34}\begin{bmatrix}13 & 8 & 1\\ 5 & -10 & 3\\ 3 & -16 & -5\end{bmatrix}$$

$$\therefore X = A^{-1}B=\frac{1}{34}\begin{bmatrix}13 & 8 & 1\\ 5 & -10 & 3\\ 3 & -6 & -5\end{bmatrix}\begin{bmatrix}1\\ \frac{3}{2}\\ 9\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{34}\begin{bmatrix}13+12+9\\ 5-15+27\\ 3-9-45\end{bmatrix}$$

$$=\frac{1}{34}\begin{bmatrix}34\\17\\-51\end{bmatrix}=\begin{bmatrix}1\\\frac{1}{2}\\-\frac{3}{2}\end{bmatrix}$$

Hence, $x=1$, $y=\frac{1}{2}$, and $z=-\frac{3}{2}$.

Question 12:

Solve system of linear equations, using matrix method.

$x-y+z=4$

$2x+y-3z=0$

$x+y+z=2$

Solution 12:

The given system of equation can be written in the form of $AX = B$,where

$$A=\begin{bmatrix}1 & -1 & 1\\2 & 1 & -3\\1 & 1 & 1\end{bmatrix}, X=\begin{bmatrix}x\\y\\z\end{bmatrix} \text{ and } B=\begin{bmatrix}4\\0\\2\end{bmatrix}.$$

Now,

$|A|=1(1+3)+1(2+3)+1(2-1)=4+5+1=10\neq 0$

Thus A is non-singular. Therefore, its inverse exists.

Now,

$A_{11}=4, A_{12}=-5, A_{13}=1$

$A_{21}=2, A_{22}=0, A_{23}=-2$

$A_{31}=2, A_{32}=5, A_{33}=3$

$$\therefore A^{-1} = \frac{1}{|A|}(adjA) = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 16+0+4 \\ -20+0+10 \\ 4+0+6 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 20 \\ -10 \\ 10 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Hence, $x = 2$, $y = -1$ and $z = 1$.

Question 13:

Solve system of linear equations, using matrix method.

$2x + 3y + 3z = 5$

$x - 2y + z = -4$

$3x - y - 2z = 3$

Solution 13:

The given system of equation can be written in the form of $AX = B$,where

$|A| = 2(4+1) - 3(2-3) + 3(-1+6)$

$= 2(5) - 3(-5) + 3(5)$

$= 10 + 15 + 15 = 40 \neq 0$

Thus, A is non-singular. Therefore, its inverse exists.

Now,

$A_{11}=5, A_{12}=5, A_{13}=5$

$A_{21}=3, A_{22}=-13, A_{23}=11$

$A_{31}=9, A_{32}=1, A_{33}=-7$

$$\therefore A^{-1}=\frac{1}{|A|}(adjA)=\frac{1}{40}\begin{bmatrix}5 & 3 & 9\\ 5 & -13 & 1\\ 5 & 11 & -7\end{bmatrix}$$

$$\therefore X=A^{-1}B=\frac{1}{40}\begin{bmatrix}5 & 3 & 9\\ 5 & -13 & 1\\ 5 & 11 & -7\end{bmatrix}\begin{bmatrix}5\\ -4\\ 3\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{40}\begin{bmatrix}25-12+27\\ 25+52+3\\ 25-44-21\end{bmatrix}$$

$$=\frac{1}{40}\begin{bmatrix}40\\ 80\\ -40\end{bmatrix}$$

$$=\begin{bmatrix}1\\ 2\\ -1\end{bmatrix}$$

Hence, $x=1$, $y=2$ and $z=-1$.

Question 14:

Solve system of linear equations, using matrix method.

$x-y+2z=7$

$3x+4y-5z=-5$

$2x-y+3z=12$

Solution 14:

The given system of equation can be written in the form of $AX=B$,where

$$A=\begin{bmatrix}1 & -1 & 2\\ 3 & 4 & -5\\ 2 & -1 & 3\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix} \text{ and } B=\begin{bmatrix}7\\ -5\\ 12\end{bmatrix}.$$

Now,

$|A| = 1(12-5) + 1(9+10) + 2(-3-8) = 7 + 19 - 22 = 4 \neq 0$

Thus, A is non-singular. Therefore, its inverse exists.

Now,

$A_{11} = 7, A_{12} = -19, A_{13} = 11$

$A_{21} = 1, A_{22} = -1, A_{23} = -1$

$A_{31} = -3, A_{32} = 11, A_{33} = 7$

$$\therefore A^{-1} = \frac{1}{|A|}(adjA) = \frac{1}{4}\begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \frac{1}{4}\begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}\begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 49-5-36 \\ -133+5+132 \\ -77+5+84 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 8 \\ 4 \\ 12 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Hence, $x = 2$, $y = 1$ and $z = 3$.

Question 15:

If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, find A^{-1} Using A^{-1} solve the system of equations

$2x - 3y + 5z = 11$

$3x + 2y - 4z = -5$

$x + y - 2z = -3$

Solution 15:

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$$

$\therefore |A| = 2(-4+4) + 3(-6+4) + 5(3-2) = 0 - 6 + 5 = -1 \neq 0$

Now, $A_{11} = 0,\ A_{12} = 2,\ A_{13} = 1$
$A_{31} = -1,\ A_{22} = -9,\ A_{23} = -5$
$A_{31} = 2,\ A_{32} = 23,\ A_{33} = 13$

$$\therefore A^{-1} = \frac{1}{|A|}(adj\ A) = -\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} \qquad ...(1)$$

Now, the given system of equations can be written in the form of $AX = B$, where

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

The solution of the system of equations is given by $X = A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}\begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix} \quad \text{[Using (1)]}$$

$$= \begin{bmatrix} 0 - 5 + 6 \\ -22 - 45 + 69 \\ -11 - 25 + 39 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Hence, $x = 1,\ y = 2,$ and $z = 3$

Question 16:

The cost of 4 Kg onion, 3kg wheat and 2kg rice is Rs 60. The cost of 2 kg onion, 4 kg wheat and 6kg rice is Rs 90. The cost of 6 kg onion 2 kg wheat and 3 kg rice is Rs 70.

Find cost of each item per kg by matrix method

Solution 16:

Let the cost of onions, wheat and rice per kg be $Rs\ X$, $Rs\ Y$ and $Rs\ Z$ respectively.

Then, the given situation can be represented by a system of equations as:

$$4x + 3y + 2z = 60$$
$$2x + 4y + 6z = 90$$
$$6x + 2y + 3z = 70$$

This system of equations can be written in the form of $AX = B$, where

$$A = \begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix}, X \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{and } B = \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}.$$

$$|A| = 4(12-12) - 3(6-36) + 2(4-24) = 0 + 90 - 40 = 50 \neq 0$$

Now,

$$A_{11} = 0, A_{12} = 30, A_{13} = -20$$
$$A_{21} = -5, A_{22} = 0, A_{23} = 10$$
$$A_{31} = 10, A_{32} = -20, A_{33} = 10$$

$$\therefore adjA = \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} adjA = \frac{1}{50} \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

Now,

$$X = A^{-1}B$$

$$\Rightarrow X = \frac{1}{50} \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix} \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{50} \begin{bmatrix} 0 - 450 + 700 \\ 1800 + 0 - 1400 \\ -1200 + 900 + 700 \end{bmatrix}$$

$$= \frac{1}{50} \begin{bmatrix} 250 \\ 400 \\ 400 \end{bmatrix}$$

$$=\begin{bmatrix}5\\8\\8\end{bmatrix}$$

$\therefore x=5, y=8,$ and $z=8$

Hence, the cost of onions is $Rs\,5$ per kg, the cost of wheat is $Rs\,8$ per kg, and the cost of rice is $Rs\,8$ per kg.

Miscellaneous Exercise

Question 1:

Prove that the determinant $\begin{vmatrix}x & \sin\theta & \cos\theta\\ -\sin\theta & -x & 1\\ \cos\theta & 1 & x\end{vmatrix}$ is independent of θ.

Solution 1:

$$\Delta=\begin{vmatrix}x & \sin\theta & \cos\theta\\ -\sin\theta & -x & 1\\ \cos\theta & 1 & x\end{vmatrix}$$

$$=x\left(x^2-1\right)-\sin\theta(-x\sin\theta-\cos\theta)+\cos\theta(-\sin\theta+x\cos\theta)$$

$$=x^3-x+x\sin^2\theta+\sin\theta\cos\theta-\sin\theta\cos\theta+x\cos^2\theta$$

$$=x^3-x+x\left(\sin^2\theta+\cos^2\theta\right)$$

$$=x^3-x+x$$

$$=x^3 \quad (\text{Independent of } \theta)$$

Hence, Δ is independent of θ.

Question 2:

Without expanding the determinant, prove that

$$\begin{vmatrix}a & a^2 & bc\\ b & b^2 & ca\\ c & c^2 & ab\end{vmatrix}=\begin{vmatrix}1 & a^2 & a^3\\ 1 & b^2 & b^3\\ 1 & c^2 & c^3\end{vmatrix}$$

Solution 2:

$$L.H.S.=\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix}$$

$$=\frac{1}{abc}\begin{vmatrix} a^2 & a^3 & abc \\ b^2 & b^3 & abc \\ c^2 & c^3 & abc \end{vmatrix} \qquad \left[R_1 \to aR_1, R_2 \to bR_2, \text{and} R_3 \to cR_3\right]$$

$$=\frac{1}{abc}.abc\begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix} \qquad \left[\text{Taking out factor } abc \ \textit{from } C_3\right]$$

$$=\begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix}$$

$$=\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} \qquad \left[\textit{Applying } C_1 \leftrightarrow C_3 \textit{and } C_2 \leftrightarrow C_3\right]$$

=R.H.S.

Hence, the given result is proved.

Question 3:

Evaluate $\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\,c\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$

Solution 3:

$$\Delta=\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\,c\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$$

Expanding along C_3, we have:

$$\Delta=-\sin\alpha\left(-\sin\alpha\sin^2\beta+\cos^2\beta\sin a\right)+\cos\alpha\left(\cos\alpha\cos^2\beta+\cos\alpha\sin^2\beta\right)$$

$$=\sin^2\alpha\left(\sin^2\beta+\cos^2\beta\right)+\cos^2\alpha\left(\cos^2\beta+\sin^2\beta\right)$$

$$=\sin^2\alpha(1)+\cos^2\alpha(1)$$

$$=1$$

Question 4:

If a, b and c are real numbers, and $\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$

Show that either $a+b+c=0$ or $a=b=c$.

Solution 4:

$$\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$$

Applying $R_1 \to R_1 + R_2 + R_3$, we have:

$$\Delta = \begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

$$= 2(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we have:

$$\Delta = 2(a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ c+a & b-c & b-a \\ a+b & c-a & c-b \end{vmatrix}$$

Expanding along R_1, we have:

$$\Delta = 2(a+b+c)(1)\left[(b-c)(c-b)-(b-a)(c-a)\right]$$

$$= 2(a+b+c)\left[-b^2 - c^2 + 2bc - bc + ba + ac - a^2\right]$$

$$= 2(a+b+c)\left[ab + bc + ca - a^2 - b^2 - c^2\right]$$

It is given that $\Delta = 0$.

$$(a+b+c)\left[ab + bc + ca - a^2 - b^2 - c^2\right] = 0$$

$\Rightarrow$ Either $a+b+c=0$,or $ab+bc+ca-a^2-b^2-c^2=0$.

Now,

$$ab+bc+ca-a^2-b^2-c^2=0$$

$$\Rightarrow -2ab-2bc-2ca+2a^3+2b^3+2c^3=0$$

$$\Rightarrow (a-b)^2+(b-c)^2+(c-a)^2=0$$

$\Rightarrow (a-b)=(b-c)^2=(c-a)^2=0$ $\left[(a-b)^2,(b-c)^2,(c-a)^2 \text{ are non-negative}\right]$

$\Rightarrow (a-b)=(b-c)=(c-a)=0$

$\Rightarrow a=b=c$

Hence, if $\Delta=0$, then either $a+b+c=0$ or $a=b=c$.

Question 5:

Solve the equations $\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix}=0, a\neq 0$

Solution 5:

$$\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix}=0$$

Applying $R_1 \to R_1+R_2+R_3$, we get:

$$\begin{vmatrix} 3x+a & 3x+a & 3x+a \\ x & x+a & x \\ x & x & x+a \end{vmatrix}=0$$

$$\Rightarrow (3x+a)\begin{vmatrix} 1 & 1 & 1 \\ x & x+a & x \\ x & x & x+a \end{vmatrix}=0$$

Applying $C_2 \to C_2-C_1$ and $C_3 \to C_3-C_1$, we have:

$$\Rightarrow (3x+a)\begin{vmatrix} 1 & 1 & 1 \\ x & a & x \\ x & x & a \end{vmatrix}=0$$

Expanding along R_1, we have:

$(3x+a)\left[1 \text{x} a^2\right]=0$

$\Rightarrow a^2(3x+a)=0$

But $a\neq 0$.

Therefore, we have:

$3x+a=0$

$\Rightarrow x = -\dfrac{a}{3}$

Question 6:

Prove that $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$

Solution 6:

$$\Delta = \begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$$

Taking out common factors a, b and c from C_1, C_2 ,and C_3, we have:

$$\Delta = abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we have:

$$\Delta = abc\begin{vmatrix} a & c & a+c \\ b & b-c & -c \\ b-a & b & -a \end{vmatrix}$$

Applying $R_2 \to R_2 + R_1$, we have:

$$\Delta = abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b-a & b & -a \end{vmatrix}$$

Applying $R_3 \to R_3 + R_2$, we have:

$$\Delta = abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ 2b & 2b & 0 \end{vmatrix}$$

$$= 2ab^2c\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ 1 & 1 & 0 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, we have:

$$\Delta = 2ab^2c \begin{vmatrix} a & c-a & a+c \\ a+b & -a & a \\ 1 & 0 & 0 \end{vmatrix}$$

Expanding along R_3, we have:

$$\Delta = 2ab^2c\left[a(c-a)+a(a+c)\right]$$

$$= 2ab^2c\left[ac-a^2+a^2+ac\right]$$

$$= 2ab^2c(2ac)$$

$$= 4a^2b^2c^2$$

Hence, the given result is proved.

Question 7:

Let $A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$ verify that

(i) $[adjA]^{-1} = adj\left(A^{-1}\right)$

(ii) $\left(A^{-1}\right)^{-1} = A$

Solution 7:

$$A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$

$$\therefore |A| = 1(15-1)+2(-10-1)+1(-2-3) = 14-22-5 = -13$$

$$A_{11} = 14, A_{12} = 11, A_{13} = -5$$

Now, $A_{21} = 11, A_{22} = 4, A_{23} = -3$

$$A_{31} = -5, A_{32} = -3, A_{33} = -1$$

$$\therefore adjA=\begin{bmatrix}14 & 11 & -5\\ 11 & 4 & -3\\ -5 & -3 & -1\end{bmatrix}$$

$$\therefore A^{-1}=\frac{1}{|A|}(adjA)$$

$$=-\frac{1}{13}\begin{bmatrix}14 & 11 & -5\\ 11 & 4 & -3\\ -5 & -3 & -1\end{bmatrix}=\frac{1}{13}\begin{bmatrix}-14 & -11 & 5\\ -11 & -4 & 3\\ 5 & 3 & 1\end{bmatrix}$$

(i)

$$|adjA|=14(-4-9)-11(-11-15)-5(-33+20)$$
$$=14(-13)-11(-26)-5(-13)$$
$$=-182+286+65=169$$

We have,

$$adj(adjA)=\begin{bmatrix}-13 & 26 & -13\\ 26 & -39 & -13\\ -13 & -13 & -65\end{bmatrix}$$

$$\therefore [adjA]^{-1}=\frac{1}{|adjA|}(adj(adjA))$$

$$=\frac{1}{169}\begin{bmatrix}-13 & 26 & -13\\ 26 & -39 & -13\\ -13 & -13 & 65\end{bmatrix}$$

$$=\frac{1}{13}\begin{bmatrix}-1 & 2 & -1\\ 2 & -3 & -1\\ -1 & -1 & -5\end{bmatrix}$$

Now, $A^{-1}=\frac{1}{13}\begin{bmatrix}-14 & -11 & 5\\ -11 & -4 & 3\\ 5 & 3 & 1\end{bmatrix}=\begin{bmatrix}-\frac{14}{13} & -\frac{11}{13} & \frac{5}{13}\\ -\frac{11}{3} & -\frac{4}{13} & \frac{3}{13}\\ \frac{5}{13} & \frac{3}{13} & \frac{1}{3}\end{bmatrix}$

$$\therefore adj\left(A^{-1}\right)=\begin{bmatrix}-\frac{4}{169}-\frac{9}{169} & -\left(-\frac{11}{169}-\frac{15}{169}\right) & -\frac{33}{169}+\frac{20}{169}\\ -\left(-\frac{11}{169}-\frac{15}{169}\right) & -\frac{14}{169}-\frac{25}{169} & -\left(-\frac{42}{169}+\frac{55}{169}\right)\\ -\frac{33}{169}+\frac{20}{169} & -\left(-\frac{42}{169}+\frac{55}{169}\right) & \frac{56}{169}-\frac{121}{169}\end{bmatrix}$$

$$=\frac{1}{169}\begin{bmatrix}-13 & 26 & -13\\ 26 & -39 & -13\\ -13 & -13 & -65\end{bmatrix}=\frac{1}{13}\begin{bmatrix}-1 & 2 & -1\\ 2 & -3 & -1\\ -1 & -1 & -5\end{bmatrix}$$

Hence, $\left[adjA\right]^{-1}=adj\left(A^{-1}\right)$.

(ii) We have shown that:

$$A^{-1}=\frac{1}{13}\begin{bmatrix}-14 & -11 & 5\\ -11 & -4 & 3\\ 5 & 3 & 1\end{bmatrix}$$

And $adjA^{-1}=\frac{1}{13}\begin{bmatrix}-1 & 2 & -1\\ 2 & -3 & -1\\ -1 & -1 & -5\end{bmatrix}$

Now,

$$\left|A^{-1}\right|=\left(\frac{1}{13}\right)^3\left[-14\times(-13)+11\times(-26)+5\times(-13)\right]$$

$$=\left(\frac{1}{13}\right)^3\times(-169)=-\frac{1}{13}$$

$$\therefore \left(A^{-1}\right) = \frac{adjA^{-1}}{\left|A^{-1}\right|} = \frac{1}{\left(-\frac{1}{13}\right)} \times \frac{1}{13}\begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & -1 \\ -1 & -1 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} = A$$

$$\therefore \left(A^{-1}\right)^{-1} = A$$

Question 8:

Evaluate $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$

Solution 8:

$$\Delta = \begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$, we have:

$$\Delta = \begin{vmatrix} 2(x+y) & 2(x+y) & 2(x+y) \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

$$= 2(x+y)\begin{vmatrix} 1 & 1 & 1 \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we have:

$$\Delta = 2(x+y)\begin{vmatrix} 1 & 0 & 0 \\ y & x & x-y \\ x+y & -y & -x \end{vmatrix}$$

Expanding along R_1, we have:

$$\Delta = 2(x+y)\left[-x^2 + y(x-y)\right]$$
$$= -2(x+y)\left(x^2 + y^2 - yx\right)$$
$$= -2\left(x^3 + y^3\right)$$

Question 9:

Evaluate $\begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$

Solution 9:

$$\Delta = \begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we have:

$$\Delta = \begin{vmatrix} 1 & x & y \\ 0 & y & 0 \\ 0 & 0 & x \end{vmatrix}$$

Expanding along C_1, we have:

$$\Delta = 1(xy - 0) = xy$$

Question 10:

Using properties of determinants, prove that:

$$\begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta & \beta^2 & \gamma+\alpha \\ \gamma & \gamma^2 & \alpha+\beta \end{vmatrix} = (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha+\beta+\gamma)$$

Solution 10:

$$\Delta = \begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta & \beta^2 & \gamma+\alpha \\ \gamma & \gamma^2 & \alpha+\beta \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we have:

$$\Delta = \begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta-\alpha & \beta^2-\alpha^2 & \alpha-\beta \\ \gamma-\alpha & \gamma^2-\alpha^2 & \alpha-\gamma \end{vmatrix}$$

$$= (\beta-\alpha)(\gamma-\alpha) \begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ 1 & \beta+\alpha & -1 \\ 1 & \gamma+\alpha & -1 \end{vmatrix}$$

Applying $R_3 \to R_3 - R_2$, we have:

$$\Delta = (\beta-\alpha)(\gamma-\alpha) \begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ 1 & \beta+\alpha & -1 \\ 0 & \gamma-\beta & 0 \end{vmatrix}$$

Expanding along R_3, we have:

$$\Delta = (\beta-\alpha)(\gamma-\alpha)\left[-(\gamma-\beta)(-\alpha-\beta-\gamma)\right]$$

$$= (\beta-\alpha)(\gamma-\alpha)(\gamma-\beta)(\alpha+\beta+\gamma)$$

$$= (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(\alpha+\beta+\gamma)$$

Hence, the given result is proved.

Question 11:

Using properties of determinants, prove that:

$$\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix} = (1+pxyz)(x-y)(y-z)(z-x)$$

Solution 11:

$$\Delta = \begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we have:

$$\Delta = \begin{vmatrix} x & x^2 & 1+px^3 \\ y-x & y^2-x^2 & p\left(y^3-x^3\right) \\ z-x & z^2-x^2 & p\left(z^3-x^3\right) \end{vmatrix}$$

$$=(y-x)(z-x)\begin{vmatrix} x & x^2 & 1+px^3 \\ 1 & y+x & p(y^2+x^2+xy) \\ 1 & z+x & p(z^2+x^2+xz) \end{vmatrix}$$

Applying $R_3 \to R_3 - R_2$, we have:

$$\Delta=(y-x)(z-x)\begin{vmatrix} x & x^2 & 1+px^3 \\ 1 & y+x & p(y^2+x^2+xy) \\ 0 & z-y & p(z-y)(x+y+z) \end{vmatrix}$$

$$=(y-x)(z-x)(z-y)\begin{vmatrix} x & x^2 & 1+px^3 \\ 1 & y+x & p(y^2+x^2+xy) \\ 0 & 1 & p(x+y+z) \end{vmatrix}$$

Expanding along R_3, we have:

$$\Delta=(x-y)(y-z)(z-x)[(-1)(p)(xy^2+x^3+x^2y)$$
$$+1+px^3+p(x+y+z)(xy)]$$
$$=(x-y)(y-z)(z-x)[-pxy^2-px^3-px^2y$$
$$+1+px^3+px^2y+pxy^2+pxyz]$$
$$=(x-y)(y-z)(z-x)(1+pxyz)$$

Hence, the given result is proved.

Question 12:

Using properties of determinants, prove that:

$$\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix}=3(a+b+c)(ab+ba+ca)$$

Solution 12:

$$\Delta=\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$. We have:

$$\Delta = \begin{vmatrix} a+b+c & -a+b & -a+c \\ a+b+c & 3b & -b+c \\ a+b+c & -c+b & 3c \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & -a+b & -a+c \\ 1 & 3b & -b+c \\ 1 & -c+b & 3c \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we have:

$$\Delta = (a+b+c)\begin{vmatrix} 1 & -a+b & -a+c \\ 0 & 2b+a & a-b \\ 0 & a-c & 2c+a \end{vmatrix}$$

Expanding along C_1, we have:

$$\Delta = (a+b+c)\left[(2b+a)(2c+a)-(a-b)(a-c)\right]$$

$$= (a+b+c)\left[4bc+2ab+2ac+a^2-a^2+ac+ba-bc\right]$$

$$= (a+b+c)(3ab+3bc+3ac)$$

$$= 3(a+b+c)(ab+bc+ca)$$

Hence, the given result is proved.

Question 13:

Using properties of determinants, prove this:

$$\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix} = 1$$

Solution 13:

$$\Delta = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix}$$

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$, we have:

$$\Delta = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & 2+p \\ 0 & 3 & 7+3p \end{vmatrix}$$

Applying $R_3 \to R_3 - 3R_2$, we have:

$$\Delta = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & 2+p \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along C_1, we have:

$$\Delta = 1\begin{vmatrix} 1 & 2+p \\ 0 & 1 \end{vmatrix} = 1(1-0) = 1$$

Question 14:

Using properties of determinants, prove that:

$$\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix} = 0$$

Solution 14:

$$\Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix}$$

$$= \frac{1}{\sin\delta\cos\delta}\begin{vmatrix} \sin\alpha\sin\delta & \cos\alpha\cos\delta & \cos\alpha\cos\delta - \sin\alpha\sin\delta \\ \sin\beta\sin\delta & \cos\beta\cos\delta & \cos\beta\cos\delta - \sin\beta\sin\delta \\ \sin\gamma\sin\delta & \cos\gamma\cos\delta & \cos\gamma\cos\delta - \sin\gamma\sin\delta \end{vmatrix}$$

Applying $C_1 \to +C_1 + C_3$, we have:

$$\Delta = \frac{1}{\sin\delta\cos\delta}\begin{vmatrix} \cos\alpha\cos\delta & \cos\alpha\cos\delta & \cos\alpha\cos\delta - \sin\alpha\sin\delta \\ \cos\beta\cos\delta & \cos\beta\cos\delta & \cos\beta\cos\delta - \sin\beta\sin\delta \\ \cos\gamma\cos\delta & \cos\gamma\cos\delta & \cos\gamma\cos\delta - \sin\gamma\sin\delta \end{vmatrix}$$

Here, two columns C_1 and C_2 are identical.

$\therefore \Delta = 0$

Hence, the given result is proved.

Question 15:

Solve the system of the following equations

$$\frac{2}{x}+\frac{3}{y}+\frac{10}{z}=4$$

$$\frac{4}{x}-\frac{6}{y}+\frac{5}{z}=1$$

$$\frac{6}{x}+\frac{9}{y}-\frac{20}{z}=2$$

Solution 15:

Let $\frac{1}{x}=p,\frac{1}{y}q,\frac{1}{z}=r.$

Then the given system of equations is as follows:

$$2p+3q+10r=4$$

$$4p-6q+5r=1$$

$$6p+9q+20r=2$$

This system can be written in the form of $AX=B$, where

$$A=\begin{bmatrix}2 & 3 & 10\\4 & -6 & 5\\6 & 9 & -20\end{bmatrix}, X=\begin{bmatrix}p\\q\\r\end{bmatrix} \text{ and } B=\begin{bmatrix}4\\1\\2\end{bmatrix}.$$

Now,

$$|A|=2(120-45)-3(-80-30)+10(36+36)$$

$$=150+330+720$$

$$=1200$$

Thus, A is non-singular. Therefore, its inverse exists.

Now,

$$A_{11}=75, A_{12}=110, A_{13}=72$$

$$A_{21}=150, A_{22}=100, A_{23}=0$$

$$A_{31}=75, A_{32}=30, A_{33}=-24$$

$$\therefore A^{-1}=\frac{1}{|A|}(adjA)$$

$$=\frac{1}{1200}\begin{bmatrix}75 & 150 & 75\\110 & -100 & 30\\72 & 0 & -24\end{bmatrix}$$

Now,

$X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{1200}\begin{bmatrix} 300+150+150 \\ 440-100+60 \\ 288+0-48 \end{bmatrix}$$

$$= \frac{1}{1200}\begin{bmatrix} 600 \\ 400 \\ 240 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{5} \end{bmatrix}$$

$\therefore p = \frac{1}{2}, q = \frac{1}{3}$, and $r = \frac{1}{5}$

Hence, $x = 2, y = 3$ and $z = 5$.

Question 16:

Choose the correct answer.

If $a, b, c,$ are in $A.P$., then the determinant

$$\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$$

A. 0
B. 1
C. x
D. 2x

Solution 16:

Answer: A

$$\Delta = \begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$$

$$=\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+(a+c) \\ x+4 & x+5 & x+2c \end{vmatrix} \qquad (2b=a+c \text{ as } a,b,\text{and}\, c \text{ are in } A.P)$$

Applying $R_1 \to R_1 - R_2$ and $R_3 \to R_3 - R_2$, we have:

$$\Delta=\begin{vmatrix} -1 & -1 & a-c \\ x+3 & x+4 & x+(a+c) \\ 1 & 1 & c-a \end{vmatrix}$$

Applying $R_1 \to R_1 + R_3$, we have:

$$\Delta=\begin{vmatrix} 0 & 0 & 0 \\ x+3 & x+4 & x+a+c \\ 1 & 1 & c-a \end{vmatrix}$$

Here, all the elements of the first row (R_1) are zero.
Hence, we have $\Delta = 0$.
The correct answer is A.

Question 17:
Choose the correct answer.

If X, Y, Z are non-zero real numbers, then the inverse of matrix $A=\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ is

A. $\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$

B. $xyz\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$

C. $\dfrac{1}{xyz}\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$

D. $\dfrac{1}{xyz}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Solution 17:

$$A=\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$$

$$\therefore |A| = x(yz-0) = xyz \neq 0$$

Now,

$A_{11} = yz, A_{12} = 0, A_{13} = 0$
$A_{21} = 0, A_{22} = xz, A_{23} = 0$
$A_{31} = 0, A_{32} = 0, A_{33} = xy$

$$\therefore adjA = \begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}(adjA)$$

$$= \frac{1}{xyz}\begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix}$$

$$= \begin{bmatrix} \frac{yz}{xyz} & 0 & 0 \\ 0 & \frac{xz}{xyz} & 0 \\ 0 & 0 & \frac{xy}{xyz} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{x} & 0 & 0 \\ 0 & \frac{1}{y} & 0 \\ 0 & 0 & \frac{1}{z} \end{bmatrix} = \begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$$

The correct answer is A.

Question 18:

Choose the correct answer.

Let $A = \begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$, where $x \le \theta \le 2\pi$, then

A. $Det\,(A) = 0$

B. $Det\,(A) \in (2, \infty)$

C. $Det\,(A) \in (2, 4)$

D. $Det(A) \in [2,4]$

Solution 18:

$$A = \begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$$

$$\therefore |A| = 1\left(1+\sin^2\theta\right) - \sin\theta\left(-\sin\theta + \sin\theta\right) + 1\left(\sin^2\theta + 1\right)$$

$$= 1 + \sin^2\theta + \sin^2\theta + 1$$

$$= 2 + 2\sin^2\theta$$

$$= 2\left(1+\sin^2\theta\right)$$

Now, $0 \le \theta \le 2\pi$

$$\Rightarrow 0 \le \sin\theta \le 1$$
$$\Rightarrow 0 \le \sin^2\theta \le 1$$
$$\Rightarrow 1 \le 1 + \sin^2\theta \le 2$$
$$\Rightarrow 2 \le 2(1+\sin^2\theta) \le 4$$
$$\therefore Det(A) \in [2,4]$$

Hence, the correct answer is D.