Exercise 7.1

Question 1:
Find an anti-derivative (or integral) of the function sin 2x by the method of inspection.

Solution 1:
The anti-derivative of sin 2x is a function of x whose derivative is sin 2x. It is known that,

$$\frac{d}{dx}(\cos 2x) = -2\sin 2x$$

$$\Rightarrow \sin 2x = -\frac{1}{2}\frac{d}{dx}(\cos 2x)$$

$$\therefore \sin 2x = \frac{d}{dx}\left(-\frac{1}{2}\cos 2x\right)$$

Therefore, the anti-derivative of $\sin 2x$ is $-\frac{1}{2}\cos 2x$.

Question 2:
Find an anti-derivative (or integral) of the function cos 3x by the method of inspection.

Solution 2:
The anti-derivative of cos 3x is a function of x whose derivative is cos 3x.
It is known that,

$$\frac{d}{dx}(\sin 3x) = 3\cos 3x$$

$$\Rightarrow \cos 3x = \frac{1}{3}\frac{d}{dx}(\sin 3x)$$

$$\therefore \cos 3x = \frac{d}{dx}\left(\frac{1}{3}\sin 3x\right)$$

Therefore, the anti-derivative of $\cos 3x$ is $\frac{1}{3}\sin 3x$.

Question 3:
Find an anti-derivative (or integral) of the function e^{2x} by the method of inspection.

Solution 3:
The anti–derivative of e^{2x} is the function of x whose derivative is e^{2x}
It is known that,

$$\frac{d}{dx}(e^{2x}) = 2e^{2x}$$

$$\Rightarrow e^{2x} = \frac{1}{2}\frac{d}{dx}(e^{2x})$$

$$\therefore e^{2x} = \frac{d}{dx}\left(\frac{1}{2}e^{2x}\right)$$

Therefore, the anti-derivative of e^{2x} is $\frac{1}{2}e^{2x}$.

Question 4:

Find an anti-derivative (or integral) of the function $(ax + b)^2$ by the method of inspection.

Solution 4:

The anti-derivative of $(ax + b)^2$ is the function of x whose derivative is $(ax + b)^2$.

It is known that,

$$\frac{d}{dx}(\text{ax+b})^3 = 3a(\text{ax+b})^2$$

$$\Rightarrow (\text{ax+b})^2 = \frac{1}{3a}\frac{d}{dx}(\text{ax+b})^3$$

$$\therefore (\text{ax+b})^2 = \frac{d}{dx}\left(\frac{1}{3a}(\text{ax+b})^3\right)$$

Therefore, the anti-derivative of $(\text{ax+b})^2$ is $\frac{1}{3a}(\text{ax+b})^3$.

Question 5:

Find an anti-derivative (or integral) of the function $\sin 2x - 4e^{3x}$ by the method of inspection.

Solution 5:

The anti-derivative of $\sin 2x - 4e^{3x}$ is the function of x whose derivative is $\sin 2x - 4e^{3x}$

It is known that,

$$\frac{d}{dx}\left(-\frac{1}{2}\cos 2x - \frac{4}{3}e^{3x}\right) = \sin 2x - 4e^{3x}$$

Therefore, the anti derivative of $\left(\sin 2x - 4e^{3x}\right)$ is $\left(-\frac{1}{2}\cos 2x - \frac{4}{3}e^{3x}\right)$.

Question 6:

$$\int\left(4e^{3x} + 1\right)dx$$

Solution 6:

$$\int\left(4e^{3x} + 1\right)dx$$

$$= 4\int e^{3x}dx + \int 1dx$$

$$= 4\left(\frac{e^{3x}}{3}\right) + x + C$$

$$= \frac{4}{3}e^{3x} + x + C$$

where C is an arbitrary constant.

Question 7:

$$\int x^2\left(1-\frac{1}{x^2}\right)dx$$

Solution 7:

$$\int x^2\left(1-\frac{1}{x^2}\right)dx$$

$$= \int\left(x^2 - 1\right)dx$$

$$= \int x^2 dx - \int 1 dx$$

$$= \frac{x^3}{3} - x + C$$

where C is an arbitrary constant.

Question 8:

$$\int\left(ax^2 + bx + c\right)dx$$

Solution 8:

$$\int\left(ax^2 + bx + c\right)dx$$

$$= a\int x^2 dx + b\int x dx + c\int 1.dx$$

$$= a\left(\frac{x^3}{3}\right) + b\left(\frac{x^2}{2}\right) + cx + C$$

$$= \frac{ax^3}{3} + \frac{bx^2}{2} + cx + C$$

where C is an arbitrary constant.

Question 9:

$$\int\left(2x^2 + e^x\right)dx$$

Solution 9:

$$\int\left(2x^2 + e^x\right)dx$$

$$=2\int x^2dx+\int e^xdx$$
$$=2\left(\frac{x^3}{3}\right)+e^x+C$$
$$=\frac{2}{3}x^3+e^x+C$$
where C is an arbitrary constant.

Question 10:

$$\int\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2dx$$

Solution 10:

$$\int\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2dx$$
$$=\int\left(x+\frac{1}{x}-2\right)dx$$
$$=\int xdx+\int\frac{1}{x}dx-2\int 1.dx$$
$$=\frac{x^2}{2}+\log|x|-2x+C$$
where C is an arbitrary constant.

Question 11:

$$\int\frac{x^3+5x^2-4}{x^2}dx$$

Solution 11:

$$\int\frac{x^3+5x^2-4}{x^2}dx$$
$$=\int\left(x+5-4x^{-2}\right)dx$$
$$=\int xdx+5\int 1.dx-4\int x^{-2}dx$$
$$=\frac{x^2}{2}+5x-4\left(\frac{x^{-1}}{-1}\right)+C$$
$$=\frac{x^2}{2}+5x+\frac{4}{x}+C$$
where C is an arbitrary constant.

Question 12:

$$\int \frac{x^3+3x+4}{\sqrt{x}}dx$$

Solution 12:

$$\int \frac{x^3+3x+4}{\sqrt{x}}dx$$

$$=\int\left(x^{\frac{5}{2}}+3x^{\frac{1}{2}}+4x^{-\frac{1}{2}}\right)dx$$

$$\frac{x^{\frac{7}{2}}}{\frac{7}{2}}+\frac{3\left(x^{\frac{3}{2}}\right)}{\frac{3}{2}}+\frac{4\left(x^{\frac{1}{2}}\right)}{\frac{1}{2}}+C$$

$$=\frac{2}{7}x^{\frac{7}{2}}+2x^{\frac{3}{2}}+8x^{\frac{1}{2}}+C$$

$$=\frac{2}{7}x^{\frac{7}{2}}+2x^{\frac{3}{2}}+8\sqrt{x}+C$$

where C is an arbitrary constant.

Question 13:

$$\int \frac{x^3-x^2+x-1}{x-1}dx$$

Solution 13:

$$\int \frac{x^3-x^2+x-1}{x-1}dx$$

On factorising, we obtain

$$\int \frac{(x^2+1)(x-1)}{x-1}dx$$

$$=\int\left(x^2+1\right)dx$$

$$=\int x^2dx+\int 1dx$$

$$=\frac{x^3}{3}+x+C$$

where C is an arbitrary constant.

Question 14:

$$\int(1-x)\sqrt{x}dx$$

Solution 14:

$$\int(1-x)\sqrt{x}dx$$
$$=\int\left(\sqrt{x}-x^{\frac{3}{2}}\right)dx$$
$$=\int x^{\frac{1}{2}}dx-\int x^{\frac{3}{2}}dx$$
$$=\frac{x^{\frac{3}{2}}}{\frac{3}{2}}-\frac{x^{\frac{5}{2}}}{\frac{5}{2}}+C$$
$$=\frac{2}{3}x^{\frac{3}{2}}-\frac{2}{5}x^{\frac{5}{2}}+C$$
where C is an arbitrary constant.

Question 15:

$$\int\sqrt{x}\left(3x^2+2x+3\right)dx$$

Solution 15:

$$\int\sqrt{x}\left(3x^2+2x+3\right)dx$$
$$=3\int x^{\frac{5}{2}}dx+2\int x^{\frac{3}{2}}dx+3\int x^{\frac{1}{2}}dx$$
$$=3\left(\frac{x^{\frac{7}{2}}}{\frac{7}{2}}\right)+2\left(\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right)+3\frac{\left(x^{\frac{3}{2}}\right)}{\frac{3}{2}}+C$$
$$=\frac{6}{7}x^{\frac{7}{2}}+\frac{4}{5}x^{\frac{5}{2}}+2x^{\frac{3}{2}}+C$$
where C is an arbitrary constant.

Question 16:

$$\int\left(2x-3\cos x+e^x\right)dx$$

Solution 16:

$$\int\left(2x-3\cos x+e^x\right)dx$$
$$=2\int xdx-3\int\cos xdx+\int e^xdx$$
$$=\frac{2x^2}{2}-3(\sin x)+e^x+C$$
$$=x^2-3\sin x+e^x+C$$
where C is an arbitrary constant.

Question 17:

$$\int\left(2x^2-3\sin x+5\sqrt{x}\right)dx$$

Solution 17:

$$\int\left(2x^2-3\sin x+5\sqrt{x}\right)dx$$

$$=2\int x^2dx-3\int \sin xdx+5\int x^{\frac{1}{2}}dx$$

$$=\frac{2x^3}{3}-3\left(-\cos x\right)+5\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right)+C$$

$$=\frac{2}{3}x^3+3\cos x+\frac{10}{3}x^{\frac{3}{2}}+C$$

where C is an arbitrary constant.

Question 18:

$$\int \sec x\left(\sec x+\tan x\right)dx$$

Solution 18:

$$\int \sec x\left(\sec x+\tan x\right)dx$$

$$=\int\left(\sec^2 x+\sec x\tan x\right)dx$$

$$=\int \sec^2 xdx+\int \sec x\tan xdx$$

$$=\tan x+\sec x+C$$

where C is an arbitrary constant.

Question 19:

$$\int\frac{\sec^2 x}{\operatorname{cosec}^2 x}dx$$

Solution 19:

$$\int\frac{\sec^2 x}{\operatorname{cosec}^2 x}dx$$

$$=\int\frac{\frac{1}{\cos^2 x}}{\frac{1}{\sin^2 x}}dx$$

$$=\int\frac{\sin^2 x}{\cos^2 x}dx$$

$= \int \tan^2 x dx$

$= \int \left(\sec^2 x - 1\right) dx$

$= \int \sec^2 x dx - \int 1 dx$

$= \tan x - x + C$

where C is an arbitrary constant.

Question 20:

$\int \frac{2 - 3\sin x}{\cos^2 x} dx$

Solution 20:

$\int \frac{2 - 3\sin x}{\cos^2 x} dx$

$= \int \left(\frac{2}{\cos^2 x} - \frac{3\sin x}{\cos^2 x} \right) dx$

$= \int 2\sec^2 x dx - 3\int \tan x \sec x dx$

$= 2\tan x - 3\sec x + C$

where C is an arbitrary constant.

Question 21:

The anti-derivative of $\left(\sqrt{x} + \frac{1}{\sqrt{x}} \right)$ equals

$(A) \frac{1}{3} x^{\frac{1}{3}} + 2x^{\frac{1}{2}} + C$ $(B) \frac{2}{3} x^{\frac{2}{3}} + \frac{1}{2} x^2 + C$

$(C) \frac{2}{3} x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$ $(D) \frac{3}{2} x^{\frac{3}{2}} + \frac{1}{2} x^{\frac{1}{2}} + C$

Solution 21:

$\int \sqrt{x} + \frac{1}{\sqrt{x}} dx$

$= \int x^{\frac{1}{2}} dx + \int x^{-\frac{1}{2}} dx$

$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C$

$= \frac{2}{3} x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$, where C is an arbitrary constant.

Hence, the correct Answer is C.

Question 22:

If $\frac{d}{dx}f(x)=4x^3-\frac{3}{x^4}$ such that f(2) = 0, then f(x) is

$(A)\,x^4+\frac{1}{x^3}-\frac{129}{8}$ $(B)\,x^3+\frac{1}{x^4}+\frac{129}{8}$

$(C)\,x^4+\frac{1}{x^3}+\frac{129}{8}$ $(D)\,x^3+\frac{1}{x^4}-\frac{129}{8}$

Solution 22:

It is given that, $\frac{d}{dx}f(x)=4x^3-\frac{3}{x^4}$

Anti-derivative of $4x^3-\frac{3}{x^4}=f(x)$

$$\therefore f(x)=\int 4x^3-\frac{3}{x^4}dx$$

$$f(x)=4\int x^3dx-3\int\left(x^{-4}\right)dx$$

$$f(x)=4\left(\frac{x^4}{4}\right)-3\left(\frac{x^{-3}}{-3}\right)+C$$

$$f(x)=x^4+\frac{1}{x^3}+C$$

Also,

f(2) = 0

$$\therefore f(2)=(2)^4+\frac{1}{(2)^3}+C=0$$

$$\Rightarrow 16+\frac{1}{8}+C=0$$

$$\Rightarrow C=-\left(16+\frac{1}{8}\right)$$

$$\Rightarrow C=\frac{-129}{8}$$

$$\therefore f(x)=x^4+\frac{1}{x^3}-\frac{129}{8}$$

Hence, the correct Answer is A.

Exercise 7.2

Question 1:

Integrate $\frac{2x}{1+x^2}$

Solution 1:

Let $1+x^2$ = t

$\therefore$ 2x dx = dt

$$\Rightarrow \int \frac{2x}{1+x^2} dx = \int \frac{1}{t} dt$$

$$= \log|t| + C$$

$$= \log\left|1+x^2\right| + C$$

$$= \log\left(1+x^2\right) + C$$

where C is an arbitrary constant.

Question 2:

Integrate $\frac{(\log x)^2}{x}$

Solution 2:

Let log x = t

$$\therefore \frac{1}{x} dx = dt$$

$$\Rightarrow \int \frac{(\log|x|)^2}{x} dx = \int t^2 dt$$

$$= \frac{t^3}{3} + C$$

$$= \frac{(\log|x|)^3}{3} + C$$

where C is an arbitrary constant.

Question 3:

Integrate $\frac{1}{x + x\log x}$

Solution 3:

The given function can be rewritten as

$$\frac{1}{x + x\log x} = \frac{1}{x(1+\log x)}$$

Let 1 + log x = t

$$\therefore \frac{1}{x} dx = dt$$

$$\Rightarrow \int \frac{1}{x(1+\log x)} dx = \int \frac{1}{t} dt$$

$$= \log|t| + C$$

$$= \log|1+\log x| + C$$

where C is an arbitrary constant.

Question 4:

Integrate sin x . sin (cos x)

Solution 4:

Let cos x = t

∴ −sin x dx = dt

$$\Rightarrow \int \sin x . \sin(\cos x) dx = -\int \sin t dt$$

$$= -[-\cos t] + C$$

$$= \cos t + C$$

$$= \cos(\cos x) + C$$

where C is an arbitrary constant.

Question 5:

Integrate sin (ax + b) cos (ax + b)

Solution 5:

The given function can be rewritten as

sin (ax + b) cos (ax + b) = $\dfrac{2\sin(ax+b)\cos(ax+b)}{2} = \dfrac{\sin 2(ax+b)}{2}$

Let 2 (ax + b) = t

∴ 2adx = dt

$$\Rightarrow \int \frac{\sin 2(ax+b)}{2} dx = \frac{1}{2}\int \frac{\sin t \, dt}{2a}$$

$$= \frac{1}{4a}[-\cos t] + C$$

$$= \frac{-1}{4a}\cos 2(ax+b) + C$$

where C is an arbitrary constant.

Question 6:

Integrate $\sqrt{ax+b}$

Solution 6:

Let ax + b = t

⇒ adx = dt

$$\therefore dx = \frac{1}{a}dt$$

$$\Rightarrow \int (ax+b)^{\frac{1}{2}}dx = \frac{1}{a}\int t^{\frac{1}{2}}dt$$

$$= \frac{1}{a}\left(\frac{t^{\frac{1}{2}}}{\frac{3}{2}}\right)+C$$

$$= \frac{2}{3a}(ax+b)^{\frac{3}{2}}+C$$

where C is an arbitrary constant.

Question 7:

Integrate $x\sqrt{x+2}$

Solution 7:

Let x + 2 = t

∴ dx = dt

$$\Rightarrow \int x\sqrt{x+2}dx = \int (t-2)\sqrt{t}dt$$

$$= \int \left(t^{\frac{3}{2}} - 2t^{\frac{1}{2}}\right)dt$$

$$= \int t^{\frac{3}{2}}dt - 2\int t^{\frac{1}{2}}dt$$

$$= \frac{t^{\frac{5}{2}}}{\frac{5}{2}} - 2\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right)+C$$

$$= \frac{2}{5}t^{\frac{5}{2}} - \frac{4}{3}t^{\frac{3}{2}}+C$$

$$= \frac{2}{5}(x+2)^{\frac{5}{2}} - \frac{4}{3}(x+2)^{\frac{3}{2}}+C$$

where C is an arbitrary constant.

Question 8:

$x\sqrt{1+2x^2}$

Solution 8:

Let $1 + 2x^2 = t$

$\therefore$ 4x dx = dt

$$\Rightarrow \int x\sqrt{1+2x^2}\,dx = \int \frac{\sqrt{t}}{4}dt$$

$$= \frac{1}{4}\int t^{\frac{1}{2}}dt$$

$$= \frac{1}{4}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{1}{6}\left(1+2x^2\right)^{\frac{3}{2}} + C$$

where C is an arbitrary constant.

Question 9:

Integrate $(4x+2)\sqrt{x^2+x+1}$

Solution 9:

Let $x^2 + x + 1 = t$

$\therefore$ (2x +1)dx = dt

$$\int (4x+2)\sqrt{x^2+x+1}\,dx$$

$$= \int 2\sqrt{t}dt$$

$$= 2\int \sqrt{t}dt$$

$$= 2\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{4}{3}\left(x^2+x+1\right)^{\frac{3}{2}} + C$$

where C is an arbitrary constant.

Question 10:

Integrate $\dfrac{1}{x-\sqrt{x}}$

Solution 10:

The given function can be rewritten as

$$\frac{1}{x-\sqrt{x}} = \frac{1}{\sqrt{x}\left(\sqrt{x}-1\right)}$$

Let $\left(\sqrt{x}-1\right)$ = t

$\therefore \frac{1}{2\sqrt{x}}dx = dt$

$\Rightarrow \int \frac{1}{\sqrt{x}\left(\sqrt{x}-1\right)}dx = \int \frac{2}{t}dt$

$= 2\log|t| + C$

$= 2\log\left|\sqrt{x}-1\right| + C$

where C is an arbitrary constant.

Question 11:

Integrate $\frac{x}{\sqrt{x+4}}, x > 0$

Solution 11:

Let x + 4 = t

$\therefore$ dx = dt

$\int \frac{x}{\sqrt{x+4}}dx = \int \frac{(t-4)}{\sqrt{t}}dt$

$= \int \left(\sqrt{t} - \frac{4}{\sqrt{t}}\right)dt$

$= \frac{t^{\frac{3}{2}}}{\frac{3}{2}} - 4\left(\frac{t^{\frac{1}{2}}}{\frac{1}{2}}\right) + C$

$= \frac{2}{3}(t)^{\frac{3}{2}} - 8(t)^{\frac{1}{2}} + C$

$= \frac{2}{3}t.t^{\frac{1}{2}} - 8t^{\frac{1}{2}} + C$

$= \frac{2}{3}t^{\frac{1}{2}}(t-12) + C$

$= \frac{2}{3}(x+4)^{\frac{1}{2}}(x+4-12) + C$

$= \frac{2}{3}\sqrt{x+4}(x-8) + C$

where C is an arbitrary constant.

Question 12:

Integrate $\left(x^3-1\right)^{\frac{1}{3}} x^5$

Solution 12:

Let $x^3 - 1 = t$

$\therefore 3x^2 dx = dt$

$\Rightarrow \int \left(x^3-1\right)^{\frac{1}{3}} x^5 dx = \int \left(x^3-1\right)^{\frac{1}{3}} x^3 . x^2 dx$

$= \int t^{\frac{1}{3}} (t+1) \frac{dt}{3}$

$= \frac{1}{3} \int \left(t^{\frac{4}{3}} + t^{\frac{1}{3}} \right) dt$

$= \frac{1}{3} \left[\frac{t^{\frac{7}{3}}}{\frac{7}{3}} + \frac{t^{\frac{4}{3}}}{\frac{4}{3}} \right] + C$

$= \frac{1}{3} \left[\frac{3}{7} t^{\frac{7}{3}} + \frac{3}{4} t^{\frac{4}{3}} \right] + C$

$= \frac{1}{7} \left(x^3-1\right)^{\frac{7}{3}} + \frac{1}{4} \left(x^3-1\right)^{\frac{4}{3}} + C$

where C is an arbitrary constant.

Question 13:

Integrate $\frac{x^2}{\left(2+3x^3\right)^3}$

Solution 13:

Let $2 + 3x^3 = t$

$\therefore 9x^2 dx = dt$

$\Rightarrow \int \frac{x^2}{\left(2+3x^3\right)^3} dx = \frac{1}{9} \int \frac{dt}{(t)^3}$

$= \frac{1}{9} \left[\frac{t^{-2}}{-2} \right] + C$

$= \frac{-1}{18} \left(\frac{1}{t^2} \right) + C$

$= \frac{-1}{18\left(2+3x^3\right)^2} + C$

where C is an arbitrary constant.

Question 14:

Integrate $\dfrac{1}{x(\log x)^m}, x > 0$

Solution 14:

Let log x = t

$$\frac{1}{x}dx = dt$$

$$\Rightarrow \int \frac{1}{x(\log x)^m}dx = \int \frac{dt}{(t)^m}$$

$$= \frac{t^{-m+1}}{-m+1} + C$$

$$= \frac{(\log x)^{1-m}}{(1-m)} + C$$

where C is an arbitrary constant.

Question 15:

Integrate $\dfrac{x}{9-4x^2}$

Solution 15:

Let $9 – 4x^2 = t$

$\therefore$ −8x dx = dt

$$\Rightarrow \int \frac{x}{9-4x^2}dx = \frac{-1}{8}\int \frac{1}{t}dt$$

$$= \frac{-1}{8}\log|t| + C$$

$$= \frac{-1}{8}\log\left|9-4x^2\right| + C$$

where C is an arbitrary constant.

Question 16:

Integrate e^{2x+3}

Solution 16:

Let 2x + 3 = t

$\therefore$ 2dx = dt

$$\Rightarrow \int e^{2x+3}dx = \frac{1}{2}\int e^t dt$$

$$=\frac{1}{2}\left(e^{t}\right)+C$$
$$=\frac{1}{2}e^{(2x+3)}+C$$
where C is an arbitrary constant.

Question 17:

Integrate $\frac{x}{e^{x^2}}$

Solution 17:

Let $x^2 = t$

$\therefore$ 2x dx = dt

$$\Rightarrow \int \frac{x}{e^{x^2}}dx=\frac{1}{2}\int \frac{1}{e^{t}}dt$$
$$=\frac{1}{2}\int e^{-t}dt$$
$$=\frac{1}{2}\left(\frac{e^{-t}}{-1}\right)+C$$
$$=-\frac{1}{2}e^{-x^2}+C$$
$$=\frac{-1}{2e^{x^2}}+C$$
where C is an arbitrary constant.

Question 18:

Integrate $\frac{e^{\tan^{-1}x}}{1+x^2}$

Solution 18:

Let $\tan^{-1}x=t$

$$\therefore \frac{1}{1+x^2}dx=dt$$
$$\Rightarrow \int \frac{e^{\tan^{-1}x}}{1+x^2}dx=\int e^{t}dt$$
$$=e^{t}+C$$
$$=e^{\tan^{-1}x}+C$$
where C is an arbitrary constant.

Question 19:

Integrate $\frac{e^{2x}-1}{e^{2x}+1}$

Solution 19:

Dividing the given function's numerator and denominator by e^x, we obtain,

$$\frac{\frac{\left(e^{2x}-1\right)}{e^x}}{\frac{\left(e^{2x}+1\right)}{e^x}}=\frac{e^x-e^{-x}}{e^x+e^{-x}}$$

Let e^x+e^{-x} = t

$$\left(e^x-e^{-x}\right)dx=dt$$

$$\Rightarrow \int\frac{e^{2x}-1}{e^{2x}+1}dx=\int\frac{e^x-e^{-x}}{e^x+e^{-x}}dx$$

$$=\int\frac{dt}{t}$$

$$=\log|t|+C$$

$$=\log\left|e^x+e^{-x}\right|+C$$

where C is an arbitrary constant.

Question 20:

Integrate $\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$

Solution 20:

Let $e^{2x}+e^{-2x}=t$

$$\Rightarrow 2e^{2x}-2e^{-2x}dx=dt$$

$$\Rightarrow 2\left(e^{2x}-e^{-2x}\right)dx=dt$$

$$\Rightarrow \int\left(\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}\right)dx=\int\frac{dt}{2t}$$

$$=\frac{1}{2}\int\frac{1}{t}dt$$

$$=\frac{1}{2}\log|t|+C$$

$$=\frac{1}{2}\log\left|e^{2x}+e^{-2x}\right|+C$$

where C is an arbitrary constant.

Question 21:

Integrate $\tan^2(2x-3)$

Solution 21:

$\tan^2(2x-3)=\sec^2(2x-3)-1$

Let 2x − 3 = t

$\therefore$ 2dx = dt

$$\Rightarrow \int \tan^2(2x-3)dx=\int\left[\sec^2(2x-3)-1\right]dx$$

$$=\frac{1}{2}\int\left(\sec^2 t\right)dt-\int 1dx$$

$$=\frac{1}{2}\int \sec^2 tdt-\int 1dx$$

$$=\frac{1}{2}\tan t-x+C$$

$$=\frac{1}{2}\tan(2x-3)-x+C$$

where C is an arbitrary constant.

Question 22:

Integrate $\sec^2(7-4x)$

Solution 22:

Let 7 − 4x = t

$\therefore$ −4dx = dt

$$\therefore \int \sec^2(7-4x)dx=\frac{-1}{4}\int \sec^2 tdt$$

$$=\frac{-1}{4}(\tan t)+C$$

$$=\frac{-1}{4}\tan(7-4x)+C$$

where C is an arbitrary constant.

Question 23:

Integrate $\frac{\sin^{-1}x}{\sqrt{1-x^2}}$

Solution 23:

Let $\sin^{-1}$ x = t

$$\frac{1}{\sqrt{1-x^2}}dx=dt$$

$$\Rightarrow \int \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx = \int t \, dt$$

$$= \frac{t^2}{2} + C$$

$$= \frac{\left(\sin^{-1} x\right)^2}{2} + C$$

where C is an arbitrary constant.

Question 24:

Integrate $\dfrac{2\cos x - 3\sin x}{6\cos x + 4\sin x}$

Solution 24:

The given function is,

$$\frac{2\cos x - 3\sin x}{6\cos x + 4\sin x} = \frac{2\cos x - 3\sin x}{2(3\cos x + 2\sin x)}$$

Let $3\cos x + 2\sin x = t$

$$(-3\sin x + 2\cos x)dx = dt$$

$$\int \frac{2\cos x - 3\sin x}{6\cos x + 4\sin x} dx = \int \frac{dt}{2t}$$

$$= \frac{1}{2}\int \frac{1}{t} dt$$

$$= \frac{1}{2}\log|t| + C$$

$$= \frac{1}{2}\log|2\sin x + 3\cos x| + C$$

where C is an arbitrary constant.

Question 25:

Integrate $\dfrac{1}{\cos^2 x(1-\tan x)^2}$

Solution 25:

The given function is

$$\frac{1}{\cos^2 x(1-\tan x)^2} = \frac{\sec^2}{(1-\tan x)^2}$$

Let $(1-\tan x) = t$

$$-\sec^2 x dx = dt$$

$$\Rightarrow \int \frac{\sec^2}{(1-\tan x)^2} dx = \int \frac{-dt}{t^2}$$

$$= -\int t^{-2} dt$$

$$= +\frac{1}{t} + C$$

$$= \frac{1}{(1-\tan x)} + C$$

where C is an arbitrary constant.

Question 26:

Integrate $\frac{\cos\sqrt{x}}{\sqrt{x}}$

Solution 26:

Let $\sqrt{\text{x}} = t$

$$\frac{1}{2\sqrt{\text{x}}} dx = dt$$

$$\Rightarrow \int \frac{\cos\sqrt{x}}{\sqrt{x}} dx = 2\int \cos t dt$$

$$= 2\sin t + C$$

$$= 2\sin\sqrt{x} + C$$

where C is an arbitrary constant.

Question 27:

Integrate $\sqrt{\sin 2x}\cos 2x$

Solution 27:

Let sin 2x = t

So, 2cos 2x dx = dt

$$\Rightarrow \int \sqrt{\sin 2x}\cos 2x dx = \frac{1}{2}\int \sqrt{t} dt$$

$$= \frac{1}{2}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{1}{3} t^{\frac{3}{2}} + C$$

$$= \frac{1}{3}(\sin 2x)^{\frac{3}{2}} + C$$

where C is an arbitrary constant.

Question 28:

Integrate $\frac{\cos x}{\sqrt{1+\sin x}}$

Solution 28:

Let 1 + sin x = t

$\therefore$ cos x dx = dt

$\Rightarrow \int \frac{\cos x}{\sqrt{1+\sin x}} dx = \int \frac{dt}{\sqrt{t}}$

$= \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C$

$= 2\sqrt{t} + C$

$= 2\sqrt{1+\sin x} + C$

where C is an arbitrary constant.

Question 29:

Integrate $\cot x \log \sin x$

Solution 29:

Let log sin x =t

$\Rightarrow \frac{1}{\text{sinx}}.\cos x dx = dt$

$\therefore \cot x dx = dt$

$\Rightarrow \int \cot x \log \sin x dx = \int t dt$

$= \frac{t^2}{2} + C$

$= \frac{1}{2}(\log \sin x)^2 + C$

where C is an arbitrary constant.

Question 30:

Integrate $\frac{\sin x}{1+\cos x}$

Solution 30:

Let 1 + cos x = t

$\therefore$ −sin x dx = dt

$$\Rightarrow \int \frac{\sin x}{1+\cos x}dx = \int -\frac{dt}{t}$$

$$= -\log|t| + C$$

$$= -\log|1+\cos x| + C$$

where C is an arbitrary constant.

Question 31:

Integrate $\dfrac{\sin x}{(1+\cos x)^2}$

Solution 31:

Let 1 + cos x = t

$\therefore$ −sin x dx = dt

$$\Rightarrow \int \frac{\sin x}{(1+\cos x)^2}dx = \int -\frac{dt}{t^2}$$

$$= -\int t^{-2}dt$$

$$= \frac{1}{t} + C$$

$$= \frac{1}{1+\cos x} + C$$

where C is an arbitrary constant.

Question 32:

Integrate $\dfrac{1}{1+\cot x}$

Solution 32:

Let $I = \int \frac{1}{1+\cot x}dx$

$$= \int \frac{1}{1+\frac{\cos x}{\sin x}}dx$$

$$= \int \frac{\sin x}{\sin x + \cos x}dx$$

$$= \frac{1}{2}\int \frac{2\sin x}{\sin x + \cos x}dx$$

$$= \frac{1}{2}\int \frac{(\sin x + \cos x) + (\sin x - \cos x)}{(\sin x + \cos x)}dx$$

$$=\frac{1}{2}\int 1dx+\frac{1}{2}\int\frac{\sin x-\cos x}{\sin x+\cos x}dx$$

$$=\frac{1}{2}(x)+\frac{1}{2}\int\frac{\sin x-\cos x}{\sin x+\cos x}dx$$

Let sin x + cos x = t ⇒ (cos x − sin x) dx = dt

$$\therefore I=\frac{x}{2}+\frac{1}{2}\int\frac{-(dt)}{t}$$

$$=\frac{x}{2}-\frac{1}{2}\log|t|+C$$

$$=\frac{x}{2}-\frac{1}{2}\log|\sin x+\cos x|+C$$

where C is an arbitrary constant.

Question 33:

Integrate $\frac{1}{1-\tan x}$

Solution 33:

Let $I=\int\frac{1}{1-\tan x}dx$

$$=\int\frac{1}{1-\frac{\sin x}{\cos x}}dx$$

$$=\int\frac{\cos x}{\cos x-\sin x}dx$$

$$=\frac{1}{2}\int\frac{2\cos x}{\cos x-\sin x}dx$$

$$=\frac{1}{2}\int\frac{(\cos x-\sin x)+(\cos x+\sin x)}{(\cos x-\sin x)}dx$$

$$=\frac{1}{2}\int 1dx+\frac{1}{2}\int\frac{\cos x+\sin x}{\cos x-\sin x}dx$$

$$=\frac{x}{2}+\frac{1}{2}\int\frac{\cos x+\sin x}{\cos x-\sin x}dx$$

Put cos x − sin x = t ⇒ (−sin x − cos x) dx = dt

$$\therefore I=\frac{x}{2}+\frac{1}{2}\int\frac{-(dt)}{t}$$

$$=\frac{x}{2}-\frac{1}{2}\log|t|+C$$

$$=\frac{x}{2}-\frac{1}{2}\log|\cos x-\sin x|+C$$

where C is an arbitrary constant.

Question 34:

Integrate $\dfrac{\sqrt{\tan x}}{\sin x \cos x}$

Solution 34:

Let $I = \int \dfrac{\sqrt{\tan x}}{\sin x \cos x} dx$

$= \int \dfrac{\sqrt{\tan x} \times \cos x}{\sin x \cos x \times \cos x} dx$

$= \int \dfrac{\sqrt{\tan x}}{\tan x \cos^2 x} dx$

$= \int \dfrac{\sec^2 x dx}{\sqrt{\tan x}}$

Let tan x = t $\Rightarrow$ $\sec^2 x$ dx = dt

$\therefore I = \int \dfrac{dt}{\sqrt{t}}$

$= 2\sqrt{t} + C$

$= 2\sqrt{\tan x + C}$

where C is an arbitrary constant.

Question 35:

Integrate $\dfrac{(1+\log x)^2}{x}$

Solution 35:

Let 1 + log x = t

$\therefore \dfrac{1}{x} dx = dt$

$\Rightarrow \int \dfrac{(1+\log x)^2}{x} dx = \int t^2 dt$

$= \dfrac{t^3}{3} + C$

$= \dfrac{(1+\log x)^3}{3} + C$

where C is an arbitrary constant.

Question 36:

Integrate $\dfrac{(x+1)(x+\log x)^2}{x}$

Solution 36:

The given function can be rewritten as

$$\frac{(x+1)(x+\log x)^2}{x}$$

$$=\left(1+\frac{1}{x}\right)(x+\log x)^2$$

Let $(x+\log x)=t$

$$\therefore \left(1+\frac{1}{x}\right)dx=dt$$

$$\Rightarrow \int\left(1+\frac{1}{x}\right)(x+\log x)^2\,dx=\int t^2 dt$$

$$=\frac{t^3}{3}+C$$

$$=\frac{1}{3}(x+\log x)^3+C$$

where C is an arbitrary constant.

Question 37:

Integrate $\dfrac{x^3\sin\left(\tan^{-1}x^4\right)}{1+x^8}$

Solution 37:

Let $x^4 = t$

$\therefore\ 4x^3\ dx = dt$

$$\Rightarrow \int\frac{x^3\sin\left(\tan^{-1}x^4\right)}{1+x^8}dx=\frac{1}{4}\int\frac{\sin\left(\tan^{-1}t\right)}{1+t^2}dt \qquad ...(1)$$

Let $\tan^{-1}t=u$

$$\therefore \frac{1}{1+t^2}dt=du$$

From (1), we obtain

$$\int\frac{x^3\sin\left(\tan^{-1}x^4\right)dx}{1+x^8}=\frac{1}{4}\int\sin u\,du$$

$$=\frac{1}{4}(-\cos u)+C$$

$$=-\frac{1}{4}\cos\left(\tan^{-1}t\right)+C$$

$$=\frac{-1}{4}\cos\left(\tan^{-1}x^4\right)+C$$

where C is an arbitrary constant.

Question 38:

$\int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x} dx$ equals

(A) $10^x - x^{10} + C$ (B) $10^x + x^{10} + C$

(C) $(10^x - x^{10})^{-1} + C$ (D) $\log(10^x + x^{10}) + C$

Solution 38:

Let $x^{10} + 10^x = t$

$\therefore (10x^9 + 10^x \log_e 10)dx = \int \frac{dt}{t}$

$\Rightarrow \int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10x} dx = \int \frac{dt}{t}$

$= \log t + C$

$= \log(10^x + x^{10}) + C$

Hence, the correct Answer is D.

Question 39:

$\int \frac{dx}{\sin^2 x \cos^2 x}$ equals

(A) $\tan x + \cot x + C$ (B) $\tan x - \cot x + C$

(C) $\tan x \cot x + C$ (D) $\tan x - \cot 2x + C$

Solution 39:

Let $I = \int \frac{dx}{\sin^2 x \cos^2 x}$

$= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx$

$= \int \frac{\sin^2 x}{\sin^2 x \cos^2 x} dx + \int \frac{\cos^2 x}{\sin^2 x \cos^2 x} dx$

$= \int \sec^2 x dx + \int \cos ec^2 dx$

$= \tan x - \cot x + C$

Hence, the correct Answer is B.

Exercise 7.3

Question 1:
Integrate $\sin^2(2x + 5)$

Solution 1:
The given function can be rewritten as

$$\sin^2(2x+5)=\frac{1-\cos 2(2x+5)}{2}=\frac{1-\cos(4x+10)}{2}$$

$$\Rightarrow \int \sin^2(2x+5)dx=\int\frac{1-\cos(4x+10)}{2}dx$$

$$=\frac{1}{2}\int 1dx-\frac{1}{2}\int\cos(4x+10)dx$$

$$=\frac{1}{2}x-\frac{1}{2}\left(\frac{\sin(4x+10)}{4}\right)+C`$$

$$=\frac{1}{2}x-\frac{1}{8}\sin(4x+10)+C$$

Question 2:
Integrate sin3x.cos4x

Solution 2:
It is known that, $\sin A\cos B=\frac{1}{2}\{\sin(A+B)+\sin(A-B)\}$

$$\therefore \int \sin 3x\cos 4x dx=\frac{1}{2}\int\{\sin(3x+4x)+\sin(3x-4x)\}dx$$

$$=\frac{1}{2}\int\{\sin 7x+\sin(-x)\}dx$$

$$=\frac{1}{2}\int\{\sin 7x-\sin x\}dx$$

$$=\frac{1}{2}\int\sin 7xdx-\frac{1}{2}\int\sin xdx$$

$$=\frac{1}{2}\left(\frac{-\cos 7x}{7}\right)-\frac{1}{2}(-\cos x)+C$$

$$=\frac{-\cos 7x}{14}+\frac{\cos x}{2}+C$$

where C is an arbitrary constant.

Question 3:
Integrate cos 2x cos 4x cos 6x

Solution 3:

It is known that, $\cos A \cos B = \frac{1}{2}\left\{\cos\left(A+B\right)+\cos(A-B\right\}$

$$\therefore \int \cos 2x\left(\cos 4x \cos 6x\right)dx = \int \cos 2x\left[\frac{1}{2}\left\{\cos\left(4x+6x\right)+\cos\left(4x-6x\right)\right\}\right]dx$$

$$= \frac{1}{2}\int\left\{\cos 2x \cos 10x + \cos 2x \cos\left(-2x\right)\right\}dx$$

$$= \frac{1}{2}\int\left\{\cos 2x \cos 10x + \cos^2 2x\right\}dx$$

$$= \frac{1}{2}\int\left[\left\{\frac{1}{2}\cos\left(2x+10x\right)+\cos\left(2x-10x\right)\right\}+\left(\frac{1+\cos 4x}{2}\right)\right]dx$$

$$= \frac{1}{4}\int\left(\cos 12x + \cos 8x + 1 + \cos 4x\right)dx$$

$$= \frac{1}{4}\left[\frac{\sin 12x}{12}+\frac{\sin 8x}{8}+x+\frac{\sin 4x}{4}+C\right]$$

where C is an arbitrary constant.

Question 4:

Integrate $\sin^3\left(2x+1\right)$

Solution 4:

Let $I = \int \sin^3(2x+1)dx$

$$\Rightarrow \int \sin^3\left(2x+1\right)dx = \int \sin^2\left(2x+1\right).\sin\left(2x+1\right)dx$$

$$= \int\left(1-\cos^2\left(2x+1\right)\right)\sin\left(2x+1\right)dx$$

Let $\cos\left(2x+1\right)=t$

$$\Rightarrow -2\sin\left(2x+1\right)dx = dt$$

$$\Rightarrow \sin\left(2x+1\right)dx = \frac{-dt}{2}$$

$$\Rightarrow I = \frac{-1}{2}\int\left(1-t^2\right)dt$$

$$= \frac{-1}{2}\left\{t-\frac{t^3}{3}\right\}+C$$

$$= \frac{-1}{2}\left\{\cos\left(2x+1\right)-\frac{\cos^3\left(2x+1\right)}{3}\right\}+C$$

$$= \frac{-\cos\left(2x+1\right)}{2}+\frac{\cos^3\left(2x+1\right)}{6}+C$$

where C is an arbitrary constant.

Question 5:

Integrate $\sin^3 x\cos^3 x$

Solution 5:

Let $I=\int \sin^3 x\cos^3 x.dx$

$=\int \cos^3 x.\sin^2 x.\sin x.dx$

$=\int \cos^3 x\left(1-\cos^2 x\right)\sin x.dx$

Let cosx = t

$\Rightarrow$ -sinx.dx = dt

$\Rightarrow I=-\int t^3\left(1-t^2\right)dt$

$=-\int\left(t^3-t^5\right)dt$

$=-\left\{\frac{t^4}{4}-\frac{t^6}{6}\right\}+C$

$=-\left\{\frac{\cos^4 x}{4}-\frac{\cos^6 x}{6}\right\}+C$

$=\frac{\cos^6 x}{6}-\frac{\cos^4 x}{4}+C$

where C is an arbitrary constant.

Question 6:

Integrate $\sin x\sin 2x\sin 3x$

Solution 6:

It is known that, $\sin A\sin B=\frac{1}{2}\left\{\cos\left(A-B\right)-\cos\left(A+B\right)\right\}$

$\therefore \int \sin x\sin 2x\sin 3xdx=\int\left[\sin x.\frac{1}{2}\left\{\cos\left(2x-3x\right)-\cos\left(2x+3x\right)\right\}\right]dx$

$=\frac{1}{2}\int\left(\sin x\cos\left(-x\right)-\sin x\cos 5x\right)dx$

$=\frac{1}{2}\int\left(\sin\cos x-\sin x\cos 5x\right)dx$

$=\frac{1}{2}\int\frac{\sin 2x}{2}dx-\frac{1}{2}\int \sin x\cos 5xdx$

$=\frac{1}{4}\left[\frac{-\cos 2x}{2}\right]-\frac{1}{2}\int\frac{1}{2}\{\sin(x+5x)+\sin(x-5x)\}dx$

$=\frac{-\cos 2x}{8}-\frac{1}{4}\int\left(\sin 6x+\sin\left(-4x\right)\right)dx$

$$=\frac{-\cos 2x}{8}-\frac{1}{4}\left[\frac{-\cos 6x}{6}+\frac{\cos 4x}{4}\right]+C$$
$$=\frac{-\cos 2x}{8}-\frac{1}{8}\left[\frac{-\cos 6x}{3}+\frac{\cos 4x}{2}\right]+C$$
$$=\frac{1}{8}\left[\frac{\cos 6x}{3}+\frac{\cos 4x}{2}-\cos 2x\right]+C$$

where C is an arbitrary constant.

Question 7:

Integrate $\sin 4x \sin 8x$

Solution 7:

It is known that, $\sin A \sin B=\frac{1}{2}\cos(A-B)-\cos(A+B)$

$$\therefore \int \sin 4x \sin 8x dx=\int\left\{\frac{1}{2}\cos(4x-8x)-\cos(4x+8x)\right\}dx$$
$$=\frac{1}{2}\int(\cos(-4x)-\cos 12x)dx$$
$$=\frac{1}{2}\int(\cos 4x-\cos 12x)dx$$
$$=\frac{1}{2}\left[\frac{\sin 4x}{4}-\frac{\sin 12x}{12}\right]+\text{C}$$

where C is an arbitrary constant.

Question 8:

Integrate $\frac{1-\cos x}{1+\cos x}$

Solution 8:

Consider,

$$\frac{1-\cos x}{1+\cos x}=\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}} \qquad \left[2\sin^2\frac{x}{2}=1-\cos x \text{ and } 2\cos^2\frac{x}{2}=1+\cos x\right]$$
$$=\tan^2\frac{x}{2}$$
$$=\left(\sec^2\frac{x}{2}-1\right)$$

$$\therefore \int \frac{1-\cos x}{1+\cos x} dx = \int \left(\sec^2 \frac{x}{2} - 1 \right) dx$$

$$= \left[\frac{\tan \frac{x}{2}}{\frac{1}{2}} - x \right] + C$$

$$= 2\tan \frac{x}{2} - x + C$$

where C is an arbitrary constant.

Question 9:

Integrate $\dfrac{\cos x}{1+\cos x}$

Solution 9:

$$\frac{\cos x}{1+\cos x} = \frac{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}{2\cos^2 \frac{x}{2}} \qquad \left[\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} \text{ and } \cos x = 2\cos^2 \frac{x}{2} - 1 \right]$$

$$= \frac{1}{2}\left[1 - \tan^2 \frac{x}{2} \right]$$

$$\therefore \int \frac{\cos x}{1+\cos x} dx = \frac{1}{2} \int \left(1 - \tan^2 \frac{x}{2} \right) dx$$

$$= \frac{1}{2} \int \left(1 - \sec^2 \frac{x}{2} + 1 \right) dx$$

$$= \frac{1}{2} \int \left(2 - \sec^2 \frac{x}{2} \right) dx$$

$$= \frac{1}{2} \left[2x - \frac{\tan \frac{x}{2}}{\frac{1}{2}} \right] + C$$

$$= x - \tan \frac{x}{2} + C$$

where C is an arbitrary constant.

Question 10:

Integrate $\sin^4 x$

Solution 10:

Consider $\sin^4 x = \sin^2 x \sin^2 x$

$$=\left(\frac{1-\cos 2x}{2}\right)\left(\frac{1-\cos 2x}{2}\right)$$
$$=\frac{1}{4}(1-\cos 2x)^2$$
$$=\frac{1}{4}\left[1+\cos^2 2x-2\cos 2x\right]$$
$$=\frac{1}{4}\left[1+\left(\frac{1+\cos 4x}{2}\right)-2\cos 2x\right]$$
$$=\frac{1}{4}\left[1+\frac{1}{2}+\frac{1}{2}\cos 4x-2\cos 2x\right]$$
$$=\frac{1}{4}\left[\frac{3}{2}+\frac{1}{2}\cos 4x-2\cos 2x\right]$$
$$\therefore \int \sin^4 x dx=\frac{1}{4}\int\left[\frac{3}{2}+\frac{1}{2}\cos 4x-2\cos 2x\right]dx$$
$$=\frac{1}{4}\left[\frac{3}{2}x+\frac{1}{2}\left(\frac{\sin 4x}{4}\right)-\frac{2\sin 2x}{2}\right]+C$$
$$=\frac{1}{8}\left[3x+\frac{\sin 4x}{4}-2\sin 2x\right]+C$$
$$=\frac{3x}{8}-\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+C$$

where C is an arbitrary constant.

Question 11:

Integrate $\cos^4 2x$

Solution 11:

$$\cos^4 2x=\left(\cos^2 2x\right)^2$$
$$=\left(\frac{1+\cos 4x}{2}\right)^2$$
$$=\frac{1}{4}\left[1+\cos^2 4x+2\cos 4x\right]$$
$$=\frac{1}{4}\left[1+\left(\frac{1+\cos 8x}{2}\right)+2\cos 4x\right]$$
$$=\frac{1}{4}\left[1+\frac{1}{2}+\frac{\cos 8x}{2}+2\cos 4x\right]$$
$$=\frac{1}{4}\left[\frac{3}{2}+\frac{\cos 8x}{2}+2\cos 4x\right]$$
$$\therefore \int \cos^4 2xdx=\int\left(\frac{3}{8}+\frac{\cos 8x}{8}+\frac{\cos 4x}{2}\right)dx$$

$$=\frac{3}{8}x+\frac{\sin 8x}{64}+\frac{\sin 4x}{8}+C$$

where C is an arbitrary constant.

Question 12:

Integrate $\dfrac{\sin^2 x}{1+\cos x}$

Solution 12:

$$\frac{\sin^2 x}{1+\cos x}=\frac{\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)^2}{2\cos^2\frac{x}{2}} \qquad \left[\sin x=2\sin\frac{x}{2}\cos\frac{x}{2};\cos x=2\cos^2\frac{x}{2}-1\right]$$

$$=\frac{4\sin^2\frac{x}{2}\cos^2\frac{x}{2}}{2\cos^2\frac{x}{2}}$$

$$=2\sin^2\frac{x}{2}$$

$$=1-\cos x$$

$$\therefore\int\frac{\sin^2 x}{1+\cos x}dx=\int(1-\cos x)dx$$

$$=x-\sin x+C$$

where C is an arbitrary constant.

Question 13:

Integrate $\dfrac{\cos 2x-\cos 2\alpha}{\cos x-\cos\alpha}$

Solution 13:

Consider,

$$\frac{\cos 2x-\cos 2\alpha}{\cos x-\cos\alpha}=\frac{-2\sin\frac{2x+2\alpha}{2}\sin\frac{2x-2a}{2}}{-2\sin\frac{x+\alpha}{2}\sin\frac{x-\alpha}{2}} \qquad \left[\cos C-\cos D=-2\sin\frac{C+D}{2}\sin\frac{C-D}{2}\right]$$

$$=\frac{\sin(x+\alpha)\sin(x-\alpha)}{\sin\left(\frac{x+\alpha}{2}\right)\sin\left(\frac{x-\alpha}{2}\right)}$$

$$=\frac{\left[2\sin\left(\frac{x+\alpha}{2}\right)\cos\left(\frac{x+\alpha}{2}\right)\right]\left[2\sin\left(\frac{x-\alpha}{2}\right)\cos\left(\frac{x-\alpha}{2}\right)\right]}{\sin\left(\frac{x+\alpha}{2}\right)\sin\left(\frac{x-\alpha}{2}\right)}$$

$$=4\cos\left(\frac{x+\alpha}{2}\right)\cos\left(\frac{x-\alpha}{2}\right)$$

$$=2\left[\cos\left(\frac{x+\alpha}{2}+\frac{x-\alpha}{2}\right)+\cos\frac{x+\alpha}{2}-\frac{x-\alpha}{2}\right]$$

$$=2\left[\cos(x)+\cos\alpha\right]$$

$$=2\cos x+2\cos\alpha$$

$$\therefore\int\frac{\cos 2x-\cos 2\alpha}{\cos x-\cos\alpha}dx=\int 2\cos x+2\cos\alpha$$

$$=2\left[\sin x+x\cos\alpha\right]+C$$

where C is an arbitrary constant.

Question 14:

Integrate $\frac{\cos x-\sin x}{1+\sin 2x}$

Solution 14:

$$\frac{\cos x-\sin x}{1+\sin 2x}=\frac{\cos x-\sin x}{\left(\sin^2 x+\cos^2 x\right)+2\sin x\cos x}\qquad\left[\sin^2 x+\cos^2 x=1;\sin 2x=2\sin x\cos x\right]$$

$$=\frac{\cos x-\sin x}{(\sin x+\cos x)^2}$$

Let $\sin x+\cos x=t$

$$\therefore\left(\cos x-\sin x\right)dx=dt$$

$$\Rightarrow\int\frac{\cos x-\sin x}{1+\sin 2x}dx=\int\frac{\cos x-\sin x}{\left(\sin x+\cos x\right)^2}dx$$

$$=\int\frac{dt}{t^2}$$

$$=\int t^{-2}dt$$

$$=-t^{-1}+C$$

$$=-\frac{1}{t}+C$$

$$=\frac{-1}{\sin x+\cos x}+C$$

where C is an arbitrary constant.

Question 15:

Integrate $\tan^3 2x\sec 2x$

Solution 15:

$$\tan^3 2x\sec 2x = \tan^2 2x\tan 2x\sec 2x$$
$$= \left(\sec^2 2x - 1\right)\tan 2x\sec 2x$$
$$= \sec^2 2x.\tan 2x\sec 2x - \tan 2x\sec 2x$$
$$\therefore \int \tan^3 2x\sec 2x\,dx = \int \sec^2 2x.\tan 2x\sec 2x\,dx - \int \tan 2x\sec 2x dx$$
$$= \int \sec^2 2x.\tan 2x\sec 2x\,dx - \frac{\sec 2x}{2} + C$$

$Let \sec 2x = t$

$\therefore 2\sec 2x\tan 2x dx = dt$

$$\therefore \int \tan^3 2x\sec 2x dx = \frac{1}{2}\int t^2 dt - \frac{\sec 2x}{2} + C$$
$$= \frac{t^3}{6} - \frac{\sec 2x}{2} + C$$
$$= \frac{(\sec 2x)^3}{6} - \frac{\sec 2x}{2} + C$$

where C is an arbitrary constant.

Question 16:

Integrate $\tan^4 x$

Solution 16:

$\tan^4 x$

$= \tan^2 x.\tan^2 x$

$= \left(\sec^2 x - 1\right)\tan^2 x$

$= \sec^2 x\tan^2 x - \tan^2 x$

$= \sec^2 x\tan^2 x - \left(\sec^2 x - 1\right)$

$= \sec^2 x\tan^2 x - \sec^2 x + 1$

$$\therefore \int \tan^4 x dx = \int \sec^2 x\tan^2 x dx - \int \sec^2 x dx + \int 1 dx$$
$$= \int \sec^2 x\tan^2 x dx - \tan x + x + C \qquad ...(1)$$

Consider $\int \sec^2 x\tan^2 x dx$

$Let \tan x = 1 \Rightarrow \sec^2 x dx = dt$

$$\Rightarrow \int \sec^2 x\tan^2 x dx = \int t^2 dt = \frac{t^3}{3} = \frac{\tan^3 x}{3}$$

From equation (1), we obtain

$$\int \tan^4 x dx = \frac{1}{3}\tan^3 x - \tan x + x + C$$

where C is an arbitrary constant.

Question 17:

Integrate $\dfrac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}$

Solution 17:

$$\frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} = \frac{\sin^3 x}{\sin^2 x \cos^2 x} + \frac{\cos^3 x}{\sin^2 x \cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x} + \frac{\cos x}{\sin^2 x}$$

$$= \tan x \sec x + \cot x \cos ecx$$

$$\therefore \int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} dx = \int (\tan x \sec x + \cot x \cos ecx) dx$$

$$= \sec x - \cos ecx + C$$

where C is an arbitrary constant.

Question 18:

Integrate $\dfrac{\cos 2x + 2\sin^2 x}{\cos^2 x}$

Solution 18:

$$\frac{\cos 2x + 2\sin^2 x}{\cos^2 x}$$

$$= \frac{\cos 2x + (1 - \cos 2x)}{\cos^2 x} \qquad \left[\cos 2x = 1 - 2\sin^2 x\right]$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

$$\therefore \int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x} dx = \int \sec^2 x dx = \tan x + C$$

where C is an arbitrary constant.

Question 19:

Integrate $\dfrac{1}{\sin x \cos^3 x}$

Solution 19:

$$\frac{1}{\sin x\cos^3 x}=\frac{\sin^2 x+\cos^2 x}{\sin x\cos^3 x}$$
$$=\frac{\sin x}{\cos^3 x}+\frac{1}{\sin x\cos x}$$
$$=\tan x\sec^2 x+\frac{\dfrac{1/\cos^2 x}{\sin x\cos x}}{\cos^2 x}$$
$$=\tan x\sec^2 x+\frac{\sec^2 x}{\tan x}$$
$$\therefore\int\frac{1}{\sin x\cos^3 x}dx=\int\tan x\sec^2 xdx+\int\frac{\sec^2 x}{\tan x}dx$$
$Let\ \tan x=1\Rightarrow\sec^2 xdx=dt$
$$\Rightarrow\int\frac{1}{\sin x\cos^3 x}dx=\int tdt+\int\frac{1}{t}dt$$
$$=\frac{t^2}{2}+\log|t|+C$$
$$=\frac{1}{2}\tan^2 x+\log|\tan x|+C$$
where C is an arbitrary constant.

Question 20:

Integrate $\dfrac{\cos 2x}{(\cos x+\sin x)^2}$

Solution 20:

$$\frac{\cos 2x}{(\cos x+\sin x)^2}=\frac{\cos 2x}{\cos^2 x+\sin^2 x+2\sin x\cos x}=\frac{\cos 2x}{1+\sin 2x}$$
$$\therefore\int\frac{\cos 2x}{(\cos x+\sin x)^2}dx=\int\frac{\cos 2x}{(1+\sin 2x)}dx$$
$Let\ 1+\sin 2x=t$

$\Rightarrow 2\cos 2xdx=dt$
$$\therefore\int\frac{\cos 2x}{(\cos x+\sin x)^2}dx=\frac{1}{2}\int\frac{1}{t}dt$$
$$=\frac{1}{2}\log|t|+C$$
$$=\frac{1}{2}\log|1+\sin 2x|+C$$

$$= \frac{1}{2}\log\left|(\sin x + \cos x)^2\right| + C$$

$$= \log\left|\sin x + \cos x\right| + C$$

where C is an arbitrary constant.

Question 21:

Integrate $\sin^{-1}(\cos x)$

Solution 21:

$\sin^{-1}(\cos x)$

$Let\ \cos x = t$

$Then,\ \sin x = \sqrt{1-t^2}$

$\Rightarrow (-\sin x)dx = dt$

$dx = \dfrac{-dt}{\sin x}$

$dx = \dfrac{-dt}{\sqrt{1-t^2}}$

$$\therefore \int \sin^{-1}(\cos x)dx = \int \sin^{-1} t\left(\frac{-dt}{\sqrt{1-t^2}}\right)$$

$$= -\int \frac{\sin^{-1} t}{\sqrt{1-t^2}}dt$$

$Let\ \sin^{-1} t = u$

$\Rightarrow \dfrac{1}{\sqrt{1-t^2}}dt = du$

$$\therefore \int \sin^{-1}(\cos x)dx = -\int 4du$$

$$= -\frac{u^2}{2} + C$$

$$= \frac{-(\sin^{1} t)^2}{2} + C$$

$$= \frac{-\left[\sin^{-1}(\cos x)\right]^2}{2} + C \quad ...(1)$$

It is known that,

$\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}$

$\therefore \sin^{-1}(\cos x) = \dfrac{\pi}{2} - \cos^{-1}(\cos x) = \left(\dfrac{\pi}{2} - x\right)$

Substituting in equation (1), we obtain

$$\int \sin^{-1}(\cos x)dx = \frac{-\left[\frac{\pi}{2}-x\right]^2}{2}+C$$

$$= -\frac{1}{2}\left(\frac{\pi^2}{2}+x^2-\pi x\right)+C$$

$$= -\frac{\pi^2}{8}-\frac{x^2}{2}+\frac{1}{2}\pi x+C$$

$$= \frac{\pi x}{2}-\frac{x^2}{2}+\left(C-\frac{\pi^2}{8}\right)$$

$$= \frac{\pi x}{2}-\frac{x^2}{2}+C_1$$

Question 22:

Integrate $\frac{1}{\cos(x-a)\cos(x-b)}$

Solution 22:

$$\frac{1}{\cos(x-a)\cos(x-b)} = \frac{1}{\sin(a-b)}\left[\frac{\sin(a-b)}{\cos(x-a)\cos(x-b)}\right]$$

$$= \frac{1}{\sin(a-b)}\left[\frac{\sin\left[(x-b)-(x-a)\right]}{\cos(x-a)\cos(x-b)}\right]$$

$$= \frac{1}{\sin(a-b)}\frac{\left[\sin(x-b)\cos(x-a)-\cos(x-b)\sin(x-a)\right]}{\cos(x-a)\cos(x-b)}$$

$$= \frac{1}{\sin(a-b)}\left[\tan(x-b)-\tan(x-a)\right]$$

$$\Rightarrow \int \frac{1}{\cos(x-a)\cos(x-b)}dx = \frac{1}{\sin(a-b)}\int\left[\tan(x-b)-\tan(x-a)\right]dx$$

$$= \frac{1}{\sin(a-b)}\left[-\log\left|\cos(x-b)\right|+\log\left|\cos(x-a)\right|\right]$$

$$= \frac{1}{\sin(a-b)}\left[\log\left|\frac{\cos(x-a)}{\cos(x-b)}\right|\right]+C$$

where C is an arbitrary constant.

Question 23:

$\int \frac{\sin^2 x-\cos^2 x}{\sin^2 x\cos^2 x}dx$ is equal to

(A) tan x + cot x + C (B) tan x + cosec x + C
(C) -tan x + cot x + C (D) tan x + sec x + C

Solution 23:

$$\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx = \int \left(\frac{\sin^2 x}{\sin^2 x \cos^2 x} - \frac{\cos^2 x}{\sin^2 x \cos^2 x} \right) dx$$

$$= \int \left(\sec^2 x - \operatorname{cosec}^2 x \right) dx$$

$$= \tan x + \cot x + C$$

Hence, the correct Answer is A.

Question 24:

$\int \frac{e^x (1+x)}{\cos^2 (e^x x)} dx$ equals

(A) $-\cot (ex^x) + C$ (B) $\tan (xe^x) + C$
(C) $\tan (e^x) + C$ (D) $\cot (e^x) + C$

Solution 24:

$$\int \frac{e^x (1+x)}{\cos^2 (e^x x)} dx$$

Let $ex^x = t$

$$\Rightarrow \left(e^x . x + e^x . 1 \right) dx = dt$$

$$e^x (x+1) dx = dt$$

$$\therefore \int \frac{e^x (1+x)}{\cos^2 (e^x x)} dx = \int \frac{dt}{\cos^2 t}$$

$$= \int \sec^2 t dt$$

$$= \tan t + C$$

$$= \tan \left(e^x . x \right) + C$$

Hence, the correct Answer is B.

Exercise 7.4

Question 1:

Integrate $\frac{3x^2}{x^6 + 1}$

Solution 1:

Let $x^3 = t$

$\therefore\ 3x^2\,dx = dt$

$$\Rightarrow \int \frac{3x^2}{x^6+1}dx = \int \frac{dt}{t^2+1}$$

$= \tan^{-1} t + C$

$= \tan^{-1}(x^3) + C$

where C is an arbitrary constant.

Question 2:

Integrate $\dfrac{1}{\sqrt{1+4x^2}}$

Solution 2:

Let $2x = t$

$\therefore\ 2dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{1+4x^2}}dx = \frac{1}{2}\int \frac{dt}{\sqrt{1+t^2}}$$

$$= \frac{1}{2}\left[\log\left|t+\sqrt{t^2+1}\right|\right]+C \qquad \left[\int \frac{1}{\sqrt{x^2+a^2}}dt = \log\left|x+\sqrt{x^2+a^2}\right|\right]$$

$$= \frac{1}{2}\log\left|2x+\sqrt{4x^2+1}\right|+C$$

where C is an arbitrary constant.

Question 3:

Integrate $\dfrac{1}{\sqrt{(2-x)^2+1}}$

Solution 3:

Let $2 - x = t$

$\Rightarrow -dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{(2-x)^2+1}}dx = -\int \frac{1}{\sqrt{t^2+1}}dt$$

$$= -\log\left|t+\sqrt{t^2+1}\right|+C \qquad \left[\int \frac{1}{\sqrt{x^2+a^2}}dt = \log\left|x+\sqrt{x^2+a^2}\right|\right]$$

$$= -\log\left|2-x+\sqrt{(2-x)^2+1}\right|+C$$

$$= \log\left|\frac{1}{(2-x)+\sqrt{x^2-4x+5}}\right|+C$$

where C is an arbitrary constant.

Question 4:

Integrate $\frac{1}{\sqrt{9-25x^2}}$

Solution 4:

Let 5x = t

$\therefore$ 5dx = dt

$$\Rightarrow \int \frac{1}{\sqrt{9-25x^2}}dx = \frac{1}{5}\int \frac{1}{\sqrt{3^2-t^2}}dt$$
$$= \frac{1}{5}\sin^{-1}\left(\frac{t}{3}\right) + C$$
$$= \frac{1}{5}\sin^{-1}\left(\frac{5x}{3}\right) + C$$

where C is an arbitrary constant.

Question 5:

Integrate $\frac{3x}{1+2x^4}$

Solution 5:

$Let\ \sqrt{2}x^2 = t$

$\therefore 2\sqrt{2}xdx = dt$

$$\Rightarrow \int \frac{3x}{1+2x^4}dx = \frac{3}{2\sqrt{2}}\int \frac{dt}{1+t^2}$$
$$= \frac{3}{2\sqrt{2}}\left[\tan^{-1}t\right] + C$$
$$= \frac{3}{2\sqrt{2}}\tan^{-1}\left(\sqrt{2}x^2\right) + C$$

where C is an arbitrary constant.

Question 6:

Integrate $\frac{x^2}{1-x^6}$

Solution 6:

Let x^3 = t

$\therefore$ $3x^2$ dx = dt

$$\Rightarrow \int \frac{x^2}{1-x^6}dx = \frac{1}{3}\int \frac{dt}{1-t^2}$$
$$= \frac{1}{3}\left[\frac{1}{2}\log\left|\frac{1+t}{1-t}\right|\right] + C$$
$$= \frac{1}{6}\log\left|\frac{1+x^3}{1-x^3}\right| + C$$

where C is an arbitrary constant.

Question 7:

Integrate $\frac{x-1}{\sqrt{x^2-1}}$

Solution 7:

$$\int \frac{x-1}{\sqrt{x^2-1}}dx = \int \frac{x}{\sqrt{x^2-1}}dx - \int \frac{1}{\sqrt{x^2-1}}dx \qquad ...(1)$$

For $\int \frac{x}{\sqrt{x^2-1}}dx$, let $x^2-1 = t \Rightarrow 2xdx = dt$

$$\therefore \int \frac{x}{\sqrt{x^2-1}}dx = \frac{1}{2}\int \frac{dt}{\sqrt{t}}$$
$$= \frac{1}{2}\int t^{-\frac{1}{2}}dt$$
$$= \frac{1}{2}\left[2t^{\frac{1}{2}}\right]$$
$$= \sqrt{t}$$
$$= \sqrt{x^2-1}$$

From (1), we obtain

$$\int \frac{x-1}{\sqrt{x^2-1}}dx = \int \frac{x}{\sqrt{x^2-1}}dx - \int \frac{1}{\sqrt{x^2-1}}dx \qquad \left[\int \frac{x}{\sqrt{x^2-a^2}}dt = \log\left|x+\sqrt{x^2-a^2}\right|\right]$$
$$= \sqrt{x^2-1} - \log\left|x+\sqrt{x^2-1}\right| + C$$

where C is an arbitrary constant.

Question 8:

Integrate $\frac{x^2}{\sqrt{x^6+a^6}}$

Solution 8:

Let $x^3 = t$

$\Rightarrow 3x^2\, dx = dt$

$$\therefore \int \frac{x^2}{\sqrt{x^6+a^6}}dx = \frac{1}{3}\int \frac{dt}{\sqrt{t^2+\left(a^3\right)^2}}$$

$$= \frac{1}{3}\log\left|t+\sqrt{t^6+a^6}\right|+C$$

$$= \frac{1}{3}\log\left|x^3+\sqrt{x^6+a^6}\right|+C$$

where C is an arbitrary constant.

Question 9:

Integrate $\dfrac{\sec^2 x}{\sqrt{\tan^2 x+4}}$

Solution 9:

Let tan x = t

$\therefore \sec^2 x\, dx = dt$

$$\Rightarrow \int \frac{\sec^2 x}{\sqrt{\tan^2 x+4}}dx = \int \frac{dt}{\sqrt{t^2+2^2}}$$

$$= \log\left|t+\sqrt{t^2+4}\right|+C$$

$$= \log\left|\tan x+\sqrt{\tan^2 x+4}\right|+C$$

where C is an arbitrary constant.

Question 10:

Integrate $\dfrac{1}{\sqrt{x^2+2x+2}}$

Solution 10:

$$\int \frac{1}{\sqrt{x^2+2x+2}}dx = \int \frac{1}{\sqrt{(x+1)^2+(1)^2}}\,dx$$

Let x+1=t

$\therefore$ dx=dt

$$\Rightarrow \int \frac{1}{\sqrt{x^2+2x+2}}dx = \int \frac{1}{\sqrt{t^2+1}}\,dt$$

$$= \log\left|t+\sqrt{t^2+1}\right|+C$$

$$= \log\left|(x+1)+\sqrt{(x+1)^2+1}\right|+C$$

$$= \log\left|(x+1)+\sqrt{x^2+2x+2}\right|+C$$

where C is an arbitrary constant.

Question 11:

Integrate $\dfrac{1}{\sqrt{9x^2+6x+5}}$

Solution 11:

$$\int \frac{1}{\sqrt{9x^2+6x+5}}dx = \int \frac{1}{\sqrt{(3x+1)^2+2^2}}dx$$

$$Let\ (3x+1)=t$$

$$\therefore 3dx = dt$$

$$\int \frac{1}{\sqrt{(3x+1)^2+2^2}}dx = \frac{1}{3}\int \frac{1}{\sqrt{t^2+2^2}}dt$$

$$= \frac{1}{3}\log|t+\sqrt{t^2+2^2}|+C$$

$$= \frac{1}{3}\log|(3x+1)+\sqrt{(3x+1)^2+2^2}|+C$$

where C is an arbitrary constant.

Question 12:

Integrate $\dfrac{1}{\sqrt{7-6x-x^2}}$

Solution 12:

7 – 6x – x^2 can be written as $7-\left(x^2+6x+9-9\right)$

Therefore,

$$7-\left(x^2+6x+9-9\right)$$

$$=16-\left(x^2+6x+9\right)$$

$$=16-(x+3)^2$$

$$=(4)^2-(x+3)^2$$

$$\therefore \int \frac{1}{\sqrt{7-6x-x^2}}dx = \int \frac{1}{\sqrt{(4)^2-(x+3)^2}}dx$$

$$Let\ x+3=1$$

$$\Rightarrow dx = dt$$

$$\Rightarrow \int \frac{1}{\sqrt{(4)^2-(x+3)^2}}dx = \int \frac{1}{\sqrt{(4)^2-(t)^2}}dt$$

$$= \sin^{-1}\left(\frac{t}{4}\right)+C$$

$$= \sin^{-1}\left(\frac{x+3}{4}\right)+C$$

where C is an arbitrary constant.

Question 13:

Integrate $\frac{1}{\sqrt{(x-1)(x-2)}}$

Solution 13:

(x – 1) (x – 2) can be written as x^2-3x+2.

Therefore,

$$x^2-3x+2$$

$$= x^2-3x+\frac{9}{4}-\frac{9}{4}+2$$

$$= \left(x-\frac{3}{2}\right)^2-\frac{1}{4}$$

$$= \left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2$$

$$\therefore \int \frac{1}{\sqrt{(x-1)(x-2)}}dx = \int \frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx$$

$$Let\ x-\frac{3}{2}=t$$

$$\therefore dx = dt$$

$$\Rightarrow \int \frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx = \int \frac{1}{\sqrt{t^2-\left(\frac{1}{2}\right)^2}}dt$$

$$= \log\left|t+\sqrt{t^2-\left(\frac{1}{2}\right)^2}\right|+C$$

$$= \log\left|\left(x-\frac{3}{2}\right)+\sqrt{x^2-3x+2}\right|+C$$

where C is an arbitrary constant.

Question 14:

Integrate $\dfrac{1}{\sqrt{8+3x-x^2}}$

Solution 14:

$8+3x-x^2$ can be written as $8-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)$

Therefore,

$$8-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)$$

$$=\frac{41}{4}-\left(x-\frac{3}{2}\right)^2$$

$$\Rightarrow \int \frac{1}{\sqrt{8+3x-x^2}}dx=\int \frac{1}{\sqrt{\frac{41}{4}-\left(x-\frac{3}{2}\right)^2}}dx$$

$$Let\, x-\frac{3}{2}=t$$

$$\therefore dx=dt$$

$$\Rightarrow \int \frac{1}{\sqrt{\frac{41}{4}-\left(x-\frac{3}{2}\right)^2}}dx=\int \frac{1}{\sqrt{\left(\frac{41}{4}\right)-t^2}}dt$$

$$=\sin^{-1}\left(\frac{t}{\frac{\sqrt{41}}{2}}\right)+C$$

$$=\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{41}}{2}}\right)+C$$

$$=\sin^{-1}\left(\frac{2x-3}{\sqrt{41}}\right)+C$$

where C is an arbitrary constant.

Question 15:

Integrate $\dfrac{1}{\sqrt{(x-a)(x-b)}}$

Solution 15:

$(x-a)(x-b)$ can be written as $x^2-(a+b)x+ab$.

Therefore,

$$x^2-(a+b)x+ab$$

$$=x^2-(a+b)x+\frac{(a+b)^2}{4}-\frac{(a+b)^2}{4}+ab$$

$$=\left[x-\left(\frac{a+b}{2}\right)\right]^2-\frac{(a-b)^2}{4}$$

$$\int\frac{1}{\sqrt{(x-a)(x-b)}}dx=\int\frac{1}{\sqrt{\left[x-\left(\frac{a+b}{2}\right)\right]^2-\frac{(a-b)^2}{4}}}dx$$

$$Let\ x-\left(\frac{a+b}{2}\right)=t$$

$$\therefore dx=dt$$

$$\int\frac{1}{\sqrt{\left[x-\left(\frac{a+b}{2}\right)\right]^2-\frac{(a-b)^2}{4}}}dx=\int\frac{1}{\sqrt{t^2-\left(\frac{a-b}{2}\right)^2}}dt$$

$$=\log\left|t+\sqrt{t^2-\left(\frac{a-b}{2}\right)^2}\right|+C$$

$$=\log\left|\left\{x-\left(\frac{a+b}{2}\right)\right\}+\sqrt{(x-a)(x-b)}\right|+C$$

Question 16:

Integrate $\frac{4x+1}{\sqrt{2x^2+x-3}}$

Solution 16:

Let $2x^2 + x - 3 = t$

$\therefore (4x + 1)\, dx = dt$

$$\Rightarrow\int\frac{4x+1}{\sqrt{2x^2+x-3}}dx=\int\frac{1}{\sqrt{t}}dt$$

$$=2\sqrt{t}+C$$

$$=2\sqrt{2x^2+x-3}+C$$

where C is an arbitrary constant.

Question 17:

Integrate $\frac{x+2}{\sqrt{x^2-1}}$

Solution 17:

Let x + 2 = $A\frac{d}{dx}\left(x^2-1\right)+B$ ...(1)

$\Rightarrow x+2=A(2x)+B$

Equating the coefficients of x and constant terms on both sides, we obtain

$2A=1\Rightarrow A=\frac{1}{2}$

$B=2$

From (1), we obtain

$(x+2)=\frac{1}{2}(2x)+2$

Then, $\int\frac{x+2}{\sqrt{x^2-1}}dx=\int\frac{\frac{1}{2}(2x)+2}{\sqrt{x^2-1}}dx$

$=\frac{1}{2}\int\frac{2x}{\sqrt{x^2-1}}dx+\int\frac{2}{\sqrt{x^2-1}}dx$ ...(2)

In $\frac{1}{2}\int\frac{2x}{\sqrt{x^2-1}}dx$ let $x^2-1=t\Rightarrow 2xdx=dt$

$\frac{1}{2}\int\frac{2x}{\sqrt{x^2-1}}dx=\frac{1}{2}\int\frac{dt}{\sqrt{t}}$

$=\frac{1}{2}\left[2\sqrt{t}\right]$

$=\sqrt{t}$

$=\sqrt{x^2-1}$

Then, $\int\frac{2}{\sqrt{x^2-1}}dx=2\int\frac{1}{\sqrt{x^2-1}}dx=2\log\left|x+\sqrt{x^2-1}\right|$

From equation (2), we obtain

$\int\frac{x+2}{\sqrt{x^2-1}}dx=\sqrt{x^2-1}+2\log\left|x+\sqrt{x^2-1}\right|+C$

where C is an arbitrary constant.

Question 18:

Integrate $\frac{5x-2}{1+2x+3x^2}$

Solution 18:

Let $5x-2$ = $A\frac{d}{dx}\left(1+2x+3x^2\right)+B$

$\Rightarrow 5x-2=A(2+6x)+B$

Equating the coefficient of x and constant term on both sides, we obtain

$5=6A \Rightarrow A=\frac{5}{6}$

$2A+B=-2 \Rightarrow B=-\frac{11}{3}$

$\therefore 5x-2=\frac{5}{6}(2+6x)+\left(-\frac{11}{3}\right)$

$\Rightarrow \int \frac{5x-2}{1+2x+3x^2}dx=\int \frac{\frac{5}{6}(2+6x)-\frac{11}{3}}{1+2x+3x^2}dx$

$=\frac{5}{6}\int \frac{2+6x}{1+2x+3x^2}dx-\frac{11}{3}\int \frac{1}{1+2x+3x^2}dx$

Let $I_1=\int \frac{2+6x}{1+2x+3x^2}dx$ and $I_2=\int \frac{1}{1+2x+3x^2}dx$

$\therefore \int \frac{5x-2}{1+2x+3x^2}dx=\frac{5}{6}I_1-\frac{11}{3}I_2 \qquad ...(1)$

$I_1=\int \frac{2+6x}{1+2x+3x^2}dx$

Let $1+2x+3x^2=t$

$\Rightarrow (2+6x)dx=dt$

$\therefore I_1=\int \frac{dt}{t}$

$I_1=\log|t|$

$I_1=\log\left|1+2x+3x^2\right| \qquad ...(2)$

$I_2=\int \frac{1}{1+2x+3x^2}dx$

$1+2x+3x^2$ can be written as $1+3\left(x^2+\frac{2}{3}x\right)$

Therefore,

$1+3\left(x^2+\frac{2}{3}x\right)$

$=1+3\left(x^2+\frac{2}{3}x+\frac{1}{9}-\frac{1}{9}\right)$

$=1+3\left(x+\frac{1}{3}\right)^2-\frac{1}{3}$

$=\frac{2}{3}+3\left(x+\frac{1}{3}\right)^2$

$=3\left[\left(x+\frac{1}{3}\right)^2+\frac{2}{9}\right]$

$$= 3\left[\left(x+\frac{1}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2\right]$$

$$I_2 = \frac{1}{3}\int \frac{1}{\left[\left(x+\frac{1}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2\right]}dx$$

$$= \frac{1}{3}\left[\frac{1}{\frac{\sqrt{2}}{3}}\tan^{-1}\left(\frac{x+\frac{1}{3}}{\frac{\sqrt{2}}{3}}\right)\right]$$

$$= \frac{1}{3}\left[\frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)\right]$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) \qquad ...(3)$$

Substituting equations (2) and (3) in equation (1), we obtain

$$\int \frac{5x-2}{1+2x+3x^2}dx = \frac{5}{6}\left[\log\left|1+2x+3x^2\right|\right] - \frac{11}{3}\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)\right] + C$$

$$= \frac{5}{6}\log\left|1+2x+3x^2\right| - \frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + C$$

where C is an arbitrary constant.

Question 19:

Integrate $\frac{6x+7}{\sqrt{(x-5)(x-4)}}$

Solution 19:

$$\frac{6x+7}{\sqrt{(x-5)(x-4)}} = \frac{6x+7}{\sqrt{x^2-9x+20}}$$

Let $6x+7 = A\frac{d}{dx}\left(x^2-9x+20\right)+B$

$$\Rightarrow 6x+7 = A(2x-9)+B$$

Equating the coefficients of x and constant term, we obtain

$2A = 6 \Rightarrow A = 3$

$-9A + B = 7 \Rightarrow B = 34$

$\therefore 6x + 7 = 3(2x - 9) + 34$

$$\int \frac{6x+7}{\sqrt{x^2-9x+20}} = \int \frac{3(2x-9)+34}{\sqrt{x^2-9x+20}}dx$$

$$=3\int\frac{2x-9}{\sqrt{x^2-9x+20}}dx+34\int\frac{1}{\sqrt{x^2-9x+20}}dx$$

Let $I_1=\int\frac{2x-9}{\sqrt{x^2-9x+20}}dx$ and $I_2=\int\frac{1}{\sqrt{x^2-9x+20}}dx$

$$\therefore\int\frac{6x+7}{\sqrt{x^2-9x+20}}=3I_1+34I_2 \qquad (1)$$

Then,

$$I_1=\int\frac{2x-9}{\sqrt{x^2-9x+20}}dx$$

$Let\ x^2-9x+20=t$

$$\Rightarrow(2x-9)dx=dt$$

$$\Rightarrow I_1=\frac{dt}{\sqrt{t}}$$

$$I_1=2\sqrt{t}$$

$$I_1=2\sqrt{x^2-9x+20} \qquad ...(2)$$

and $I_2=\int\frac{1}{\sqrt{x^2-9x+20}}dx$

$x^2-9x+20$ can be written as $x^2-9x+20+\frac{81}{4}-\frac{81}{4}$.

Therefore,

$$x^2-9x+20+\frac{81}{4}-\frac{81}{4}$$

$$=\left(x-\frac{9}{2}\right)^2-\frac{1}{4}$$

$$=\left(x-\frac{9}{2}\right)^2-\left(\frac{1}{2}\right)^2$$

$$\Rightarrow I_2=\int\frac{1}{\left(x-\frac{9}{2}\right)^2-\left(\frac{1}{2}\right)^2}dx$$

$$I_2=\log\left|\left(x-\frac{9}{2}\right)+\sqrt{x^2-9x+20}\right| \qquad ...(3)$$

Substituting equations (2) and (3) in (1), we obtain

$$\int\frac{6x+7}{\sqrt{x^2-9x+20}}dx=3\left[2\sqrt{x^2-9x+20}\right]+34\log\left[\left(x-\frac{9}{2}\right)+\sqrt{x^2-9x+20}\right]+C$$

$$=6\sqrt{x^2-9x+20}+34\log\left[\left(x-\frac{9}{2}\right)+\sqrt{x^2-9x+20}\right]+C$$

where C is an arbitrary constant.

Question 20:

Integrate $\frac{x+2}{\sqrt{4x-x^2}}$

Solution 20:

Let x + 2 = $A\frac{d}{dx}\left(4x-x^2\right)+B$

$\Rightarrow x+2=A(4-2x)+B$

Equating the coefficients of x and constant term on both sides, we obtain

$-2A=1\Rightarrow A=-\frac{1}{2}$

$4A+B=2\Rightarrow B=4$

$\Rightarrow (x+2)=-\frac{1}{2}(4-2x)+4$

$$\therefore \int\frac{x+2}{\sqrt{4x-x^2}}dx=\int\frac{-\frac{1}{2}(4-2x)+4}{\sqrt{4x-x^2}}dx$$

$$=-\frac{1}{2}\int\frac{4-2x}{\sqrt{4x-x^2}}dx+4\int\frac{1}{\sqrt{4x-x^2}}dx$$

$Let\ I_1=\int\frac{4-2x}{\sqrt{4x-x^2}}dx\ and\ I_2\int\frac{1}{\sqrt{4x-x^2}}dx$

$\therefore \int\frac{x+2}{\sqrt{4x-x^2}}dx=-\frac{1}{2}I_1+4I_2$ …. (1)

Then, $I_1=\int\frac{4-2x}{\sqrt{4x-x^2}}dx$

$Let\ 4x-x^2=t$

$\Rightarrow (4-2x)dx=dt$

$\Rightarrow I_1=\int\frac{dt}{\sqrt{t}}=2\sqrt{t}=2\sqrt{4x-x^2}$ ...(2)

$I_2=\int\frac{1}{\sqrt{4x-x^2}}dx$

$\Rightarrow 4x-x^2=-\left(-4x+x^2\right)$

$=\left(-4x+x^2+4-4\right)$

$=4-(x-2)^2$

$=(2)^2-(x-2)^2$

$\therefore I_2=\int\frac{1}{\sqrt{(2)^2-(x-2)^2}}dx=\sin^{-1}\left(\frac{x-2}{2}\right)$ ...(3)

Using equations (2) and (3) in (1), we obtain

$$\int \frac{x+2}{\sqrt{4x-x^2}}dx = -\frac{1}{2}\left(2\sqrt{4x-x^2}\right) + 4\sin^{-1}\left(\frac{x-2}{2}\right) + C$$
$$= -\sqrt{4x-x^2} + 4\sin^{-1}\left(\frac{x-2}{2}\right) + C$$

Where C is an arbitrary constant

Question 21:

Integrate $\frac{x+2}{\sqrt{x^2+2x+3}}$

Solution 21:

$$\int \frac{x+2}{\sqrt{x^2+2x+3}}dx = \frac{1}{2}\int \frac{2(x+2)}{\sqrt{x^2+2x+3}}dx$$
$$= \frac{1}{2}\int \frac{2x+4}{\sqrt{x^2+2x+3}}dx$$
$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}}dx + \frac{1}{2}\int \frac{2}{\sqrt{x^2+2x+3}}dx$$
$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}}dx + \int \frac{1}{\sqrt{x^2+2x+3}}dx$$

$$Let\ I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}}dx\ and\ I_2 = \int \frac{1}{\sqrt{x^2+2x+3}}dx$$

$$\therefore \int \frac{x+2}{\sqrt{x^2+2x+3}}dx = \frac{1}{2}I_1 + I_2 \qquad ...(1)$$

Then, $I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}}dx$

Let $x^2 + 2x + 3 = t$

$\Rightarrow (2x + 2)\, dx = dt$

$$I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{x^2+2x+3} \qquad ...(2)$$

$$I_2 = \int \frac{1}{\sqrt{x^2+2x+3}}dx$$

$$\Rightarrow x^2+2x+3 = x^2+2x+1+2 = (x+1)^2 + \left(\sqrt{2}\right)^2$$

$$\therefore I_2 = \int \frac{1}{\sqrt{(x+1)^2+\left(\sqrt{2}\right)^2}}dx = \log\left|(x+1)+\sqrt{x^2+2x+3}\right| \qquad ...(3)$$

Using equations (2) and (3) in (1), we obtain

$$\int \frac{x+2}{\sqrt{x^2+2x+3}}dx = \frac{1}{2}\left[2\sqrt{x^2+2x+3}\right] + \log\left|(x+1)+\sqrt{x^2+2x+3}\right| + C$$
$$= \sqrt{x^2+2x+3} + \log\left|(x+1)+\sqrt{x^2+2x+3}\right| + C$$

Where C is an arbitrary constant

Question 22:

Integrate $\frac{x+3}{x^2-2x-5}$

Solution 22:

Let $(x+3)=A\frac{d}{dx}(x^2-2x-5)+B$

$(x+3)=A(2x-2)+B$

Equating the coefficients of x and constant term on both sides, we obtain

$2A=1\Rightarrow A=\frac{1}{2}$

$-2A+B=3\Rightarrow B=4$

$\therefore (x+3)=\frac{1}{2}(2x-2)+4$

$\Rightarrow\int\frac{x+3}{x^2-2x-5}dx=\int\frac{\frac{1}{2}(2x-2)+4}{x^2-2x-5}dx$

$=\frac{1}{2}\int\frac{2x-2}{x^2-2x-5}dx+4\int\frac{1}{x^2-2x-5}dx$

Let $I_1=\int\frac{2x-2}{x^2-2x-5}dx$ and $I_2=\int\frac{1}{x^2-2x-5}dx$

$\therefore\int\frac{x+3}{x^2-2x-5}dx=\frac{1}{2}I_1+4I_2$ ...(1)

Then, $I_1=\int\frac{2x-2}{x^2-2x-5}dx$

Let $x^2-2x-5=\text{t}$

$\Rightarrow(2x-2)dx=dt$

$\Rightarrow I_1=\int\frac{dt}{t}=\log|t|=\log|x^2-2x-5|$ ...(2)

$I_2=\int\frac{1}{x^2-2x-5}dx$

$=\int\frac{1}{(x^2-2x+1)-6}dx$

$=\int\frac{1}{(x-1)^2-(\sqrt{6})^2}dx$

$=\frac{1}{2\sqrt{6}}\log\left(\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right)$ ...(3)

Substituting (2) and (3) in (1), we obtain

$$\int \frac{x+3}{x^2-2x-5}dx = \frac{1}{2}\log\left|x^2-2x-5\right| + \frac{4}{2\sqrt{6}}\log\left|\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right| + C$$

$$= \frac{1}{2}\log\left|x^2-2x-5\right| + \frac{2}{\sqrt{6}}\log\left|\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right| + C$$

Where C is an arbitrary constant.

Question 23:

Integrate $\dfrac{5x+3}{\sqrt{x^2+4x+10}}$

Solution 23:

Let $5x+3 = A\dfrac{d}{dx}\left(x^2+4x+10\right)+B$

$\Rightarrow 5x+3 = A(2x+4)+B$

Equating the coefficients of x and constant term, we obtain

$2A = 5 \Rightarrow A = \dfrac{5}{2}$

$4A+B = 3 \Rightarrow B = -7$

$\therefore 5x+3 = \dfrac{5}{2}(2x+4)-7$

$$\Rightarrow \int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \int \frac{\frac{5}{2}(2x+4)-7}{\sqrt{x^2+4x+10}}dx$$

$$= \frac{5}{2}\int \frac{2x+4}{\sqrt{x^2+4x+10}}dx - 7\int \frac{1}{\sqrt{x^2+4x+10}}dx$$

Let $I_1 = \int \dfrac{2x+4}{\sqrt{x^2+4x+10}}dx$ and $I_2 = \int \dfrac{1}{\sqrt{x^2+4x+10}}dx$

$$\therefore \int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \frac{5}{2}I_1 - 7I_2 \qquad \text{....(1)}$$

Then, $I_1 = \int \dfrac{2x+4}{\sqrt{x^2+4x+10}}dx$

Let $x^2+4x+10$ = t

$\therefore (2x+4)dx = dt$

$$\Rightarrow I_1 = \int \frac{dt}{t} = 2\sqrt{t} = 2\sqrt{x^2+4x+10} \qquad \text{...(2)}$$

$$I_2 = \int \frac{1}{\sqrt{x^2+4x+10}}dx$$

$$= \int \frac{1}{\sqrt{\left(x^2+4x+4\right)+6}}\,dx$$

$$= \int \frac{1}{(x+2)^2+\left(\sqrt{6}\right)^2}\,dx$$

$$= \log|(x+2)+\sqrt{x^2+4x+10}| \qquad \ldots (3)$$

Using equations (2) and (3) in (1), we obtain

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}}\,dx = \frac{5}{2}\left[2\sqrt{x^2+4x+10}\right] - 7\log\left|(x+2)\sqrt{x^2+4x+10}\right| + C$$

$$= 5\sqrt{x^2+4x+10} - 7\log\left|(x+2)\sqrt{x^2+4x+10}\right| + C$$

Where C is an arbitrary constant.

Question 24:

$\int \frac{dx}{x^2+2x+2}$ equals

(A) x $\tan^{-1}$ (x + 1) + C (B) $\tan^{-1}$ (x + 1) + C

(C) (x + 1) $\tan^{-1}$ x + C (D) $\tan^{-1}$x + C

Solution 24:

$$\int \frac{dx}{x^2+2x+2} = \int \frac{dx}{\left(x^2+2x+1\right)+1}$$

$$= \int \frac{1}{(x+1)^2+(1)^2}\,dx$$

$$= \left[\tan^{-1}(x+1)\right] + C$$

Hence, the correct Answer is B.

Question 25:

$\int \frac{dx}{\sqrt{9x-4x^2}}$ equals

(A) $\frac{1}{9}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$ (B) $\frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right)+C$

(C) $\frac{1}{3}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$ (D) $\frac{1}{2}\sin^{-1}\left(\frac{9x-8}{9}\right)+C$

Solution 25:

$$\int \frac{dx}{\sqrt{9x-4x^2}}$$

$$=\int \frac{1}{\sqrt{-4\left(x^2-\frac{9}{4}x\right)}}dx$$

$$=\int \frac{1}{-4\left(x^2-\frac{9}{4}x+\frac{81}{64}-\frac{81}{64}\right)}dx$$

$$=\int \frac{1}{\sqrt{-4\left[\left(x-\frac{9}{8}\right)^2-\left(\frac{9}{8}\right)^2\right]}}dx$$

$$=\frac{1}{2}\left[\sin^{-1}\left(\frac{x-\frac{9}{8}}{\frac{9}{8}}\right)\right]+C \qquad \left(\int \frac{dy}{\sqrt{a^2-y^2}}=\sin^{-1}\frac{y}{a}+C\right)$$

$$=\frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right)+C$$

Hence, the correct Answer is B.

Exercise 7.5

Question 1:

Integrate $\frac{x}{(x+1)(x+2)}$

Solution 1:

Let $\frac{x}{(x+1)(x+2)}=\frac{A}{(x+1)}+\frac{B}{(x+2)}$

$\Rightarrow x=A(x+2)+B(x+1)$

Equating the coefficients of x and constant term, we obtain

A + B = 1

2A + B = 0

On solving, we obtain

A = −1 and B = 2

$$\therefore \frac{x}{(x+1)(x+2)}=\frac{-1}{(x+1)}+\frac{2}{(x+2)}$$

$$\Rightarrow \int \frac{x}{(x+1)(x+2)}dx=\int \frac{-1}{(x+1)}+\frac{2}{(x+2)}dx$$

$$= -\log|x+1| + 2\log|x+2| + C$$

$$= \log(x+2)^2 - \log|x+1| + C$$

$$= \log\frac{(x+2)^2}{(x+1)} + C$$

Where C is an arbitrary constant

Question 2:

Integrate $\frac{1}{x^2-9}$

Solution 2:

Let $\frac{1}{(x+3)(x-3)} = \frac{A}{(x+3)} + \frac{B}{(x-3)}$

$1 = A(x-3) + B(x+3)$

Equating the coefficients of x and constant term, we obtain

A + B = 0

−3A + 3B = 1

On solving, we obtain

$A = -\frac{1}{6}$ and $B = \frac{1}{6}$

$$\therefore \frac{1}{(x+3)(x-3)} = \frac{-1}{6(x+3)} + \frac{1}{6(x-3)}$$

$$\Rightarrow \int \frac{1}{(x^2-9)}dx = \int\left(\frac{-1}{6(x+3)} + \frac{1}{6(x-3)}\right)dx$$

$$= -\frac{1}{6}\log|x+3| + \frac{1}{6}\log|x-3| + C$$

$$= \frac{1}{6}\log\frac{|(x-3)|}{|(x+3)|} + C$$

Where C is an arbitrary constant

Question 3:

Integrate $\frac{3x-1}{(x-1)(x-2)(x-3)}$

Solution 3:

Let $\frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$

$$3x-1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \quad ...(1)$$

Equating the coefficients of x^2, x and constant term, we obtain

A + B + C = 0

– 5A – 4B – 3C = 3

6A + 3B + 2C = – 1

Solving these equations, we obtain

A = 1, B = −5, and C = 4

$$\therefore \frac{3x-1}{(x-1)(x-2)(x-3)}=\frac{1}{(x-1)}-\frac{5}{(x-2)}+\frac{4}{(x-3)}$$

$$\Rightarrow \int \frac{3x-1}{(x-1)(x-2)(x-3)}dx=\int\left\{\frac{1}{(x-1)}-\frac{5}{(x-2)}+\frac{4}{(x-3)}\right\}dx$$

$$=\log|x-1|-5\log|x-2|+4\log|x-3|+C$$

Where C is an arbitrary constant.

Question 4:

Integrate $\frac{x}{(x-1)(x-2)(x-3)}$

Solution 4:

Let $\frac{x}{(x-1)(x-2)(x-3)}=\frac{A}{(x-1)}+\frac{B}{(x-2)}+\frac{C}{(x-3)}$

$$x=A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2) \quad ...(1)$$

Equating the coefficients of x^2, x and constant term, we obtain

A + B + C = 0

–5A – 4B – 3C = 1

6A + 4B + 2C = 0

Solving these equations, we obtain

$A=\frac{1}{2}, B=2 \text{ and } C=\frac{3}{2}$

$$\therefore \frac{x}{(x-1)(x-2)(x-3)}=\frac{1}{2(x-1)}-\frac{2}{(x-2)}+\frac{3}{2(x-3)}$$

$$\Rightarrow \int \frac{x}{(x-1)(x-2)(x-3)}dx=\int\left\{\frac{1}{2(x-1)}-\frac{2}{(x-2)}+\frac{3}{2(x-3)}\right\}dx$$

$$=\frac{1}{2}\log|x-1|-2\log|x-2|+\frac{3}{2}\log|x-3|+C$$

Question 5:

Integrate $\frac{2x}{x^2+3x+2}$

Solution 5:

Let $\dfrac{2x}{x^2+3x+2}=\dfrac{A}{(x+1)}+\dfrac{B}{(x+2)}$

$2x=A(x+2)+B(x+1) \qquad ...(1)$

Equating the coefficients of x^2, x and constant term, we obtain

A + B = 2

2A + B = 0

Solving these equations, we obtain

A = −2 and B = 4

$$\therefore \frac{2x}{(x+1)(x+2)}=\frac{-2}{(x+1)}+\frac{4}{(x+2)}$$

$$\Rightarrow \int \frac{2x}{(x+1)(x+2)}dx=\int\left\{\frac{4}{(x+2)}-\frac{2}{(x+1)}\right\}dx$$

$$=4\log|x+2|-2\log|x+1|+C$$

Where C is an arbitrary constant.

Question 6:

Integrate $\dfrac{1-x^2}{x(1-2x)}$

Solution 6:

It can be seen that the given integrand is not a proper fraction.

Therefore, on dividing $(1-x^2)$ by $x(1-2x)$, we obtain

$$\frac{1-x^2}{x(1-2x)}=\frac{1}{2}+\frac{1}{2}\left(\frac{2-x}{x(1-2x)}\right)$$

Let $\dfrac{2-x}{x(1-2x)}=\dfrac{A}{x}+\dfrac{B}{(1-2x)}$

$\Rightarrow (2-x)=A(1-2x)+Bx \qquad ...(1)$

Equating the coefficients of x^2, x and constant term, we obtain

–2A + B = –1

And A = 2

Solving these equations, we obtain

A = 2 and B = 3

$$\therefore \frac{2-x}{x(1-2x)}=\frac{2}{x}+\frac{3}{1-2x}$$

Substituting in equation (1), we obtain

$$\frac{1-x^2}{x(1-2x)}=\frac{1}{2}+\frac{1}{2}\left\{\frac{2}{x}+\frac{3}{(1-2x)}\right\}$$

$$\int \frac{1-x^2}{x(1-2x)}dx = \int\left\{\frac{1}{2}+\frac{1}{2}\left(\frac{2}{x}+\frac{3}{(1-2x)}\right)\right\}dx$$

$$=\frac{x}{2}+\log|x|+\frac{3}{2(-2)}\log|1-2x|+C$$

$$=\frac{x}{2}+\log|x|-\frac{3}{4}\log|1-2x|+C$$

Where C is an arbitrary constant.

Question 7:

Integrate $\dfrac{x}{\left(x^2+1\right)\left(x-1\right)}$

Solution 7:

Let $\dfrac{x}{\left(x^2+1\right)\left(x-1\right)}=\dfrac{Ax+B}{\left(x^2+1\right)}+\dfrac{C}{\left(x-1\right)}$

$$x=\left(Ax+B\right)\left(x-1\right)+C\left(x^2+1\right)$$

$$x=Ax^2-Ax+Bx-B+Cx^2+C$$

Equating the coefficients of x^2, x, and constant term, we obtain

A + C = 0

−A + B = 1

−B + C = 0

On solving these equations, we obtain

$$A=-\frac{1}{2}, B=\frac{1}{2}, \text{ and } C=\frac{1}{2}$$

From equation (1), we obtain

$$\therefore \frac{x}{\left(x^2+1\right)\left(x-1\right)}=\frac{\left(-\frac{1}{2}x+\frac{1}{2}\right)}{x^2+1}+\frac{\frac{1}{2}}{\left(x-1\right)}$$

$$\Rightarrow \int\frac{x}{\left(x^2+1\right)\left(x-1\right)}=-\frac{1}{2}\int\frac{x}{x^2+1}dx+\frac{1}{2}\int\frac{1}{x^2+1}dx+\frac{1}{2}\int\frac{1}{x-1}dx$$

$$=-\frac{1}{4}\int\frac{2x}{x^2+1}dx+\frac{1}{2}\tan^{-1}x+\frac{1}{2}\log|x-1|+C$$

Consider $\displaystyle\int\frac{2x}{x^2+1}dx, let\ \left(x^2+1\right)=t \Rightarrow 2xdx=dt$

$$\Rightarrow \int\frac{2x}{x^2+1}dx=\int\frac{dt}{t}=\log|t|=\log\left|x^2+1\right|$$

$$\therefore \int\frac{x}{\left(x^2+1\right)\left(x-1\right)}=-\frac{1}{4}\log\left|x^2+1\right|+\frac{1}{2}\tan^{-1}x+\frac{1}{2}\log|x-1|+C$$

$$= \frac{1}{2}\log|x-1| - \frac{1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x + C$$

Where C is an arbitrary constant.

Question 8:

Integrate $\dfrac{x}{(x-1)^2(x+2)}$

Solution 8:

Let $\dfrac{x}{(x-1)^2(x+2)} = \dfrac{A}{(x-1)} + \dfrac{B}{(x-1)^2} + \dfrac{C}{(x+2)}$

$$x = A(x-1)(x+2) + B(x+2) + C(x-1)^2$$

Equating the coefficients of x^2, x and constant term, we obtain

A + C = 0

A + B – 2C = 1

−2A + 2B + C = 0

On solving, we obtain

$A = \dfrac{2}{9}$ and $C = \dfrac{-2}{9}$

$B = \dfrac{1}{3}$

$$\therefore \frac{x}{(x-1)^2(x+2)} = \frac{2}{9(x-1)} + \frac{1}{3(x-1)^2} - \frac{2}{9(x+2)}$$

$$\Rightarrow \int \frac{x}{(x-1)^2(x+2)}dx = \frac{2}{9}\int \frac{1}{(x-1)}dx + \frac{1}{3}\int \frac{1}{(x-1)^2}dx - \frac{2}{9}\int \frac{1}{(x+2)}dx$$

$$= \frac{2}{9}\log|x-1| + \frac{1}{3}\left(\frac{-1}{x-1}\right) - \frac{2}{9}\log|x+2| + C$$

$$= \frac{2}{9}\log\left|\frac{x-1}{x+2}\right| - \frac{1}{3(x-1)} + C$$

Where C is an arbitrary constant.

Question 9:

Integrate $\dfrac{3x+5}{x^3-x^2-x+1}$

Solution 9:

$$\frac{3x+5}{x^3-x^2-x+1} = \frac{3x+5}{(x-1)^2(x+1)}$$

Let $\frac{3x+5}{(x-1)^2(x+1)}=\frac{A}{(x-1)}+\frac{B}{(x-1)^2}+\frac{C}{(x+1)}$

$3x+5=A(x-1)(x+1)+B(x+1)+C(x-1)^2$

$3x+5=A(x^2-1)+B(x+1)+C(x^2+1-2x)$ ....(1)

Equating the coefficients of x^2, x and constant term, we obtain

A + C = 0

B − 2C = 3

– A + B + C = 5

On solving, we obtain

B = 4

$A=-\frac{1}{2}$ and $C=\frac{1}{2}$

$\therefore \frac{3x+5}{(x-1)^2(x+1)}=\frac{-1}{2(x-1)}+\frac{4}{(x-1)^2}+\frac{1}{2(x+1)}$

$\Rightarrow \int\frac{3x+5}{(x-1)^2(x+1)}dx=-\frac{1}{2}\int\frac{1}{x-1}dx+4\int\frac{1}{(x-1)^2}dx+\frac{1}{2}\int\frac{1}{(x+1)}dx$

$=-\frac{1}{2}\log|x-1|+4\left(\frac{-1}{x-1}\right)+\frac{1}{2}\log|x+1|+C$

$=\frac{1}{2}\log\left|\frac{x+1}{x-1}\right|-\frac{4}{(x-1)}+C$

Where C is an arbitrary constant.

Question 10:

Integrate $\frac{2x-3}{(x^2-1)(2x+3)}$

Solution 10:

$\frac{2x-3}{(x^2-1)(2x+3)}=\frac{2x-3}{(x+1)(x-1)(2x+3)}$

Let $\frac{2x-3}{(x+1)(x-1)(2x+3)}=\frac{A}{(x+1)}+\frac{B}{(x-1)}+\frac{C}{(2x+3)}$

$\Rightarrow (2x-3)=A(x-1)(2x+3)+B(x+1)(2x+3)+C(x+1)(x-1)$

$\Rightarrow (2x-3)=A(2x^2+x-3)+B(2x^2+5x+3)+C(x^2-1)$

$\Rightarrow (2x-3)=(2A+2B+C)x^2+(A+5B)x+(-3A+3B-C)$

Equating the coefficients of x^2, x and constant, we obtain

2A + 2B + C = 0

A + 5B = 2

– 3A + 3B – C = – 3

On solving, we obtain

$B=-\frac{1}{10}, A=\frac{5}{2},$ and C=-$\frac{24}{5}$

$$\therefore \frac{2x-3}{(x+1)(x-1)(2x+3)} = \frac{5}{2(x+1)} - \frac{1}{10(x-1)} - \frac{24}{5(2x+3)}$$

$$\Rightarrow \int \frac{2x-3}{(x^2-1)(2x+3)} dx = \frac{5}{2}\int \frac{1}{(x+1)} dx - \frac{1}{10}\int \frac{1}{x-1} dx - \frac{24}{5}\int \frac{1}{(2x+3)} dx$$

$$= \frac{5}{2}\log|x+1| - \frac{1}{10}\log|x-1| - \frac{24}{5\times 2}\log|2x+3|$$

$$= \frac{5}{2}\log|x+1| - \frac{1}{10}\log|x-1| - \frac{12}{5}\log|2x+3| + C$$

Where C is an arbitrary constant.

Question 11:

Integrate $\frac{5x}{(x+1)(x^2-4)}$

Solution 11:

$$\frac{5x}{(x+1)(x^2-4)} = \frac{5x}{(x+1)(x+2)(x-2)}$$

Let $\frac{5x}{(x+1)(x+2)(x-2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)} + \frac{C}{(x-2)}$

$$5x = A(x+2)(x-2) + B(x+1)(x-2) + C(x+1)(x+2) \quad \ldots(1)$$

Equating the coefficients of x^2, x and constant, we obtain

A + B + C = 0

– B + 3C = 5 and

– 4A – 2B + 2C = 0

On solving, we obtain

$A = \frac{5}{3}, B = -\frac{5}{2}$, and $C = \frac{5}{6}$

$$\therefore \frac{5x}{(x+1)(x+2)(x-2)} = \frac{5}{3(x+1)} + -\frac{5}{2(x+2)} + \frac{5}{6(x-2)}$$

$$\Rightarrow \int \frac{5x}{(x+1)(x^2-4)} dx = \frac{5}{3}\int \frac{1}{(x+1)} dx - \frac{5}{2}\int \frac{1}{(x+2)} dx + \frac{5}{6}\int \frac{1}{(x-2)} dx$$

$$= \frac{5}{3}\log|x+1| - \frac{5}{2}\log|x+2| + \frac{5}{6}\log|x-2| + C$$

Where C is an arbitrary constant.

Question 12:

Integrate $\frac{x^3+x+1}{x^2-1}$

Solution 12:

It can be seen that the given integrand is not a proper fraction.
Therefore, on dividing $(x^3 + x + 1)$ by $x^2 - 1$, we obtain

$$\frac{x^3+x+1}{x^2-1}=x+\frac{2x+1}{x^2-1}$$

Let $\frac{2x+1}{x^2-1}=\frac{A}{(x+1)}+\frac{B}{(x-1)}$

$$2x+1=A(x-1)+B(x+1) \qquad ...(1)$$

Equating the coefficients of x and constant, we obtain
$A + B = 2$
$-A + B = 1$
On solving, we obtain

$$A=\frac{1}{2} \text{ and } B=\frac{3}{2}$$

$$\therefore \frac{x^3+x+1}{x^2-1}=x+\frac{1}{2(x+1)}+\frac{3}{2(x-1)}$$

$$\Rightarrow \int\frac{x^3+x+1}{x^2+1}dx=\int xdx+\frac{1}{2}\int\frac{1}{(x+1)}dx+\frac{3}{2}\int\frac{1}{(x-1)}dx$$

$$=\frac{x^2}{2}+\log|x+1|+\frac{3}{2}\log|x-1|+C$$

Where C is an arbitrary constant.

Question 13:

Integrate $\frac{2}{(1-x)(1+x^2)}$

Solution 13:

Let $\frac{2}{(1-x)(1+x^2)}=\frac{A}{(1-x)}+\frac{Bx+C}{(1+x^2)}$

$$2=A(1+x^2)+(Bx+C)(1-x)$$

$$2=A+Ax^2+Bx-Bx^2+C-Cx$$

Equating the coefficient of x^2, x, and constant term, we obtain
$A - B = 0$
$B - C = 0$
$A + C = 2$
On solving these equations, we obtain
$A = 1, B = 1$, and $C = 1$

$$\therefore \frac{2}{(1-x)(1+x^2)}=\frac{1}{1-x}+\frac{x+1}{1+x^2}$$

$$\Rightarrow \int \frac{2}{(1-x)(1+x^2)} dx = \int \frac{1}{1-x} dx + \int \frac{x}{1+x^2} dx + \int \frac{1}{1+x^2} dx$$

$$= -\int \frac{1}{1-x} dx + \frac{1}{2}\int \frac{2x}{1+x^2} dx + \int \frac{1}{1+x^2} dx$$

$$= -\log|x-1| + \frac{1}{2}\log|1+x^2| + \tan^{-1} x + C$$

Where C is an arbitrary constant.

Question 14:

Integrate $\frac{3x-1}{(x+2)^2}$

Solution 14:

Let $\frac{3x-1}{(x+2)^2} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2}$

$\Rightarrow 3x-1 = A(x+2)+B$

Equating the coefficient of x and constant term, we obtain

A = 3

2A + B = −1 ⇒ B = −7

$$\therefore \frac{3x-1}{(x+2)^2} = \frac{3}{(x+2)} - \frac{7}{(x+2)^2}$$

$$\Rightarrow \int \frac{3x-1}{(x+2)^2} dx = 3\int \frac{1}{(x+2)} dx - 7\int \frac{1}{(x+2)^2} dx$$

$$= 3\log|x+2| - 7\left(\frac{-1}{(x+2)}\right) + C$$

$$= 3\log|x+2| + \frac{7}{(x+2)} + C$$

Where C is an arbitrary constant.

Question 15:

Integrate $\frac{1}{x^4-1}$

Solution 15:

$$\frac{1}{(x^4-1)} = \frac{1}{(x^2-1)(x^2+1)} = \frac{1}{(x+1)(x-1)(1+x^2)}$$

Let $\frac{1}{(x+1)(x-1)(1+x^2)} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{Cx+D}{(x^2+1)}$

$$1 = A(x-1)(1+x^2) + B(x+1)(1+x^2) + (Cx+D)(x^2-1)$$

$$1 = A(x^3 + x - x^2 - 1) + B(x^3 + x + x^2 + 1) + Cx^3 + Dx^2 - Cx - D$$

$$1 = (A+B+C)x^3 + (-A+B+D)x^2 + (A+B-C)x + (-A+B-D)$$

Equating the coefficient of x^3, x^2, x, and constant term, we obtain

$$A+B+C=0$$

$$-A+B+D=0$$

$$A+B-C=0$$

$$-A+B-D=1$$

$$A=-\frac{1}{4}, B=\frac{1}{4}, C=0, \textit{ and } D=-\frac{1}{2}$$

$$\therefore \frac{1}{(x^4-1)} = \frac{-1}{4(x+1)} + \frac{1}{4(x-1)} + \frac{1}{2(x^2+1)}$$

$$\Rightarrow \int \frac{1}{x^4-1}dx = -\frac{1}{4}\log|x-1| + \frac{1}{4}\log|x-1| - \frac{1}{2}\tan^{1} x + C$$

$$= \frac{1}{4}\log\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\tan^{1} x + C$$

Where C is an arbitrary constant.

Question 16:

Integrate $\dfrac{1}{x(x^n+1)}$

[*Hint*: multiply numerator and denominator by x^{n-1} and put $x^n = t$]

Solution 16:

$$\frac{1}{x(x^n+1)}$$

Multiplying numerator and denominator by x^{n-1}, we obtain

$$\frac{1}{x(x^n+1)} = \frac{x^{n-1}}{x^{n-1}x(x^n+1)} = \frac{x^{n-1}}{x^n(x^n+1)}$$

Let $x^n = t \Rightarrow n\, x^{n-1}\, dx = dt$

$$\therefore \int \frac{1}{x(x^n+1)}dx = \int \frac{x^{n-1}}{x^n(x^n+1)}dx = \frac{1}{n}\int \frac{1}{t(t+1)}dt$$

$$\textit{Let } \frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{(t+1)}$$

$$1 = A(1+t) + Bt \qquad ...(1)$$

Equating the coefficients of t and constant, we obtain

A = 1 and B = −1

$$\therefore \frac{1}{t(t+1)}=\frac{1}{t}-\frac{1}{(1+t)}$$

$$\Rightarrow \int \frac{1}{x(x^n+1)}dx=\frac{1}{n}\int\left\{\frac{1}{t}-\frac{1}{(1+t)}\right\}dx$$

$$=\frac{1}{n}\left[\log|t|-\log|t+1|\right]+C$$

$$=\frac{1}{n}\left[\log|x^n|-\log|x^n+1|\right]+C$$

$$=\frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right|+C$$

Where C is an arbitrary constant.

Question 17:

Integrate $\dfrac{\cos x}{(1-\sin x)(2-\sin x)}$

[Hint: Put sin x = t]

Solution 17:

$$\frac{\cos x}{(1-\sin x)(2-\sin x)}$$

Let $\sin x=t \Rightarrow \cos x dx=dt$

$$\therefore \int \frac{\cos x}{(1-\sin x)(2-\sin x)}dx=\int \frac{dt}{(1-t)(2-t)}$$

Let $\dfrac{1}{(1-t)(2-t)}=\dfrac{A}{(1-t)}+\dfrac{B}{(2-t)}$

$1=A(2-t)+B(1-t)$ ...(1)

Equating the coefficients of t and constant, we obtain

– A – B = 0 and

2A + B = 1

On solving, we obtain

A = 1 and B = -1

$$\therefore \frac{1}{(1-t)(2-t)}=\frac{1}{(1-t)}-\frac{1}{(2-t)}$$

$$\Rightarrow \int \frac{\cos x}{(1-\sin x)(2-\sin x)}dx=\int\left\{\frac{1}{1-t}-\frac{1}{(2-t)}\right\}dt$$

$$=-\log\lfloor 1-t\rfloor+\log|2-t|+C$$

$$=\log\left|\frac{2-t}{1-t}\right|+C$$

$$= \log\left|\frac{2-\sin x}{1-\sin x}\right| + C$$

Where C is an arbitrary constant.

Question 18:

Integrate $\dfrac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$

Solution 18:

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = 1 - \frac{4x^2+10}{(x^2+3)(x^2+4)}$$

Let $\dfrac{(4x^2+10)}{(x^2+3)(x^2+4)} = \dfrac{Ax+B}{(x^2+3)} + \dfrac{Cx+D}{(x^2+4)}$

$$4x^2+10 = (Ax+B)(x^2+4) + (Cx+D)(x^2+3)$$

$$4x^2+10 = Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + 3Cx + Dx^2 + 3D$$

$$4x^2+10 = (A+C)x^3 + (B+D)x^2 + (4A+3C)x + (4B+3D)$$

Equating the coefficients of x^3, x^2, x and constant term, we obtain

A + C = 0

B + D = 4

4A + 3C = 0

4B + 3D = 10

On solving these equations, we obtain

A = 0, B = −2, C = 0, and D = 6

$$\therefore \frac{(4x^2+10)}{(x^2+3)(x^2+4)} = \frac{-2}{(x^2+3)} + \frac{6}{(x^2+4)}$$

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = 1 - \left(\frac{-2}{(x^2+3)} + \frac{6}{(x^2+4)}\right)$$

$$\Rightarrow \int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}dx = \int \left\{1 + \frac{2}{(x^2+3)} - \frac{6}{(x^2+4)}\right\}dx$$

$$= \int \left\{1 + \frac{2}{x^2+\left(\sqrt{3}\right)^2} - \frac{6}{x^2+2^2}\right\}$$

$$= x + 2\left(\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}\right) - 6\left(\frac{1}{2}\tan^{-1}\frac{x}{2}\right) + C$$

$$= x + \frac{2}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} - 3\tan^{-1}\frac{x}{2} + C$$

Question 19:

Integrate $\frac{2x}{\left(x^2+1\right)\left(x^2+3\right)}$

Solution 19:

$$\frac{2x}{\left(x^2+1\right)\left(x^2+3\right)}$$

Let $x^2 = t \Rightarrow 2x\,dx = dt$

$$\therefore \int \frac{2x}{\left(x^2+1\right)\left(x^2+3\right)}dx = \int \frac{dt}{(t+1)(t+3)} \qquad ...(1)$$

Let $\frac{1}{(t+1)(t+3)} = \frac{A}{(t+1)} + \frac{B}{(t+3)}$

$$1 = A(t+3) + B(t+1) \qquad ...(2)$$

Equating the coefficients of t and constant, we obtain

A + B = 0 and 3A + B = 1

On solving, we obtain

$A = \frac{1}{2}$ and $B = -\frac{1}{2}$

$$\therefore \frac{1}{(t+1)(t+3)} = \frac{1}{2(t+1)} - \frac{1}{2(t+3)}$$

$$\Rightarrow \int \frac{2x}{\left(x^2+1\right)\left(x^2+3\right)}dx = \int \left\{\frac{1}{2(t+1)} - \frac{1}{2(t+3)}\right\}dt$$

$$= \frac{1}{2}\log\left|(t+1)\right| - \frac{1}{2}\log\left|t+3\right| + C$$

$$= \frac{1}{2}\log\left|\frac{t+1}{t+3}\right| + C$$

$$= \frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right| + C$$

Where C is an arbitrary constant.

Question 20:

Integrate $\frac{1}{x\left(x^4-1\right)}$

Solution 20:

$$\frac{1}{x\left(x^4-1\right)}$$

Multiplying numerator and denominator by x^3, we obtain

$$\frac{1}{x\left(x^4-1\right)}=\frac{x^3}{x^4\left(x^4-1\right)}$$

$$\therefore \int \frac{1}{x\left(x^4-1\right)}dx=\int \frac{x^3}{x^4\left(x^4-1\right)}dx$$

Let $x^4 = t \Rightarrow 4x^3\, dx = dt$

$$\therefore \int \frac{1}{x\left(x^4-1\right)}dx=\frac{1}{4}\int \frac{dt}{t\left(t-1\right)}$$

Let $\frac{1}{t\left(t-1\right)}=\frac{A}{t}+\frac{B}{\left(t-1\right)}$

$$1=A\left(t-1\right)+Bt \qquad ...(1)$$

Equating the coefficients of t and constant, we obtain

$A + B = 0$ and $-A = 1$

$A = -1$ and $B = 1$

$$\Rightarrow \frac{1}{t\left(t-1\right)}=\frac{-1}{t}+\frac{1}{t-1}$$

$$\Rightarrow \int \frac{1}{x\left(x^4-1\right)}dx=\frac{1}{4}\int\left\{\frac{-1}{t}+\frac{1}{t-1}\right\}dt$$

$$=\frac{1}{4}\left[-\log|t|+\log|t-1|\right]+C$$

$$=\frac{1}{4}\log\left|\frac{t-1}{t}\right|+C$$

$$=\frac{1}{4}\log\left|\frac{x^4-1}{x^4}\right|+C$$

Where C is an arbitrary constant.

Question 21:

Integrate $\frac{1}{\left(e^x-1\right)}$

[Hint: Put $e^x = t$]

Solution 21:

Let $e^x = t \Rightarrow e^x\, dx = dt$

$$\Rightarrow \int \frac{1}{\left(e^x-1\right)}dx=\int \frac{1}{t-1}\times\frac{dt}{t}=\int \frac{1}{t\left(t-1\right)}dt$$

$$Let\ \frac{1}{t\left(t-1\right)}=\frac{A}{t}+\frac{B}{t-1}$$

$$1=A\left(t-1\right)+Bt \qquad ...(1)$$

Equating the coefficients of t and constant, we obtain

A + B = 0 and –A = 1
A = −1 and B = 1

$$\therefore \frac{1}{t(t-1)} = \frac{-1}{t} + \frac{1}{t-1}$$

$$\Rightarrow \int \frac{1}{t(t-1)} dt = \log\left|\frac{t-1}{t}\right| + C$$

$$= \log\left|\frac{e^x - 1}{e^x}\right| + C$$

Where C is an arbitrary constant.

Question 22:

$\int \frac{xdx}{(x-1)(x-2)}$ equals

A. $\log\left|\frac{(\text{x-1})^2}{x-2}\right| + C$

B. $\log\left|\frac{(\text{x-2})^2}{x-1}\right| + C$

C. $\log\left|\left(\frac{x-1}{x-2}\right)^2\right| + C$

D. $\log\left|(\text{x-1})(x-2)\right| + C$

Solution 22:

Let $\frac{x}{(x-1)(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-2)}$

$x = A(x-2) + B(x-1) \qquad ...(1)$

Equating the coefficients of x and constant, we obtain
A + B = 1 and –2A – B = 0
A = −1 and B = 2

$$\therefore \frac{x}{(x-1)(x-2)} = -\frac{1}{(x-1)} + \frac{2}{(x-2)}$$

$$\Rightarrow \int \frac{x}{(x-1)(x-2)} dx = \int \left\{\frac{-1}{(x-1)} + \frac{2}{(x-2)}\right\} dx$$

$$= -\log|x-1| + 2\log|x-2| + C$$

$$= \log\left|\frac{(x-2)^2}{x-1}\right| + C$$

Hence, the correct Answer is B.

Question 23:

$\int \frac{dx}{x\left(x^2+1\right)}$ equals

A. $\log|\mathrm{x}|-\frac{1}{2}\log\left(x^2+1\right)+C$

B. $\log|\mathrm{x}|+\frac{1}{2}\log\left(x^2+1\right)+C$

$C.\ -\log|x|+\frac{1}{2}\log\left(x^2+1\right)+C$

$D.\ \frac{1}{2}\log|x|+\log\left(x^2+1\right)+C$

Solution 23:

Let $\frac{1}{x\left(x^2+1\right)}=\frac{A}{x}+\frac{Bx+C}{x^2+1}$

$1=A\left(x^2+1\right)+\left(Bx+C\right)x$

Equating the coefficients of x^2, x, and constant term, we obtain

A + B = 0

C = 0

A = 1

On solving these equations, we obtain

A = 1, B = −1, and C = 0

$\therefore\ \frac{1}{x\left(x^2+1\right)}=\frac{1}{x}+\frac{-x}{x^2+1}$

$\Rightarrow\int\frac{1}{x\left(x^2+1\right)}dx=\int\left\{\frac{1}{x}-\frac{x}{x^2+1}\right\}dx$

$=\log|x|-\frac{1}{2}\log\left|x^2+1\right|+C$

Hence, the correct Answer is A.

Alter:

$\Rightarrow\int\frac{1}{x\left(x^2+1\right)}dx=\int\left\{\frac{x}{x^2(x^2+1)}\right\}dx$

Let x^2 = t, therefore, 2x dx = dt

$\therefore\int\frac{x}{x^2(x^2+1)}dx=\frac{1}{2}\int\frac{dt}{t(t+1)}=\frac{1}{2}\int\frac{(t+1)-t}{t(t+1)}dt=\frac{1}{2}\int\frac{1}{t}-\frac{1}{t+1}dt$

$=\frac{1}{2}[\log t-\log(t+1)]+C$

$=\log|x|-\frac{1}{2}\log\left|x^2+1\right|+C$

Exercise 7.6

Question 1:
Integrate x sin x

Solution 1:
Let I = $\int x \sin x dx$
Taking x as first function and sin x as second function and integrating by parts, we obtain,
I = $x \int \sin x dx - \int \left\{ \left(\frac{d}{dx} x \right) \int \sin x \, dx \right\} dx$
$= x(-\cos x) - \int 1.(-\cos x) dx$
$= -x \cos x + \sin x + C$
Where C is an arbitrary constant.

Question 2:
Integrate x sin 3x

Solution 2:
Let I = $\int x \sin 3x dx$
Taking x as first function and sin 3x as second function and integrating by parts, we obtain
$I = x \int \sin 3x dx - \int \left\{ \left(\frac{d}{dx} x \right) \int \sin 3x dx \right\}$
$= x\left(\frac{-\cos 3x}{3} \right) - \int 1.\left(\frac{-\cos 3x}{3} \right) dx$
$= \frac{-x \cos 3x}{3} + \frac{1}{3} \int \cos 3x dx$
$= \frac{-x \cos 3x}{3} + \frac{1}{9} \sin 3x + C$
Where C is an arbitrary constant.

Question 3:
Integrate $x^2 e^x$

Solution 3:
Let $I = \int x^2 e^x dx$
Taking x^2 as first function and ex as second function and integrating by parts, we obtain
$I = x^2 \int e^x dx - \int \left\{ \left(\frac{d}{dx} x^2 \right) \int e^x dx \right\} dx$

$= x^2e^x - \int 2xe^x dx$

$= x^2e^x - 2\int x \cdot e^x dx$

Again integrating by parts, we obtain

$= x^2e^x - 2\left[x \cdot \int e^x dx - \int\left\{\left(\frac{d}{dx}x\right)\int e^x dx\right\}dx\right]$

$= x^2e^x - 2\left[xe^x - \int e^x dx\right]$

$= x^2e^x - 2\left[xe^x - e^x\right]$

$= x^2e^x - 2xe^x + 2e^x + C$

$= e^x\left(x^2 - 2x + 2\right) + C$

Where C is an arbitrary constant.

Question 4:

Integrate x log x

Solution 4:

Let $I = \int x \log x dx$

Taking log x as first function and x as second function and integrating by parts, we obtain

$I = \log x \int x dx - \int\left\{\left(\frac{d}{dx}\log x\right)\int x dx\right\}dx$

$= \log x . \frac{x^2}{2} - \int \frac{1}{x} . \frac{x^2}{2} dx$

$= \frac{x^2 \log x}{2} - \int \frac{x}{2} dx$

$= \frac{x^2 \log x}{2} - \frac{x^2}{4} + C$

Where C is an arbitrary constant.

Question 5:

Integrate *x* log 2x

Solution 5:

Let $I = \int x \log 2x dx$

Taking log 2x as first function and x as second function and integrating by parts, we obtain

$I = \log 2x \int x\, dx - \int\left\{\left(\frac{d}{dx}\log 2x\right)\int x\, dx\right\}dx$

$= \log 2x . \frac{x^2}{2} - \int \frac{2}{2x} . \frac{x^2}{2} dx$

$$= \frac{x^2 \log 2x}{2} - \int \frac{x}{2} dx$$
$$= \frac{x^2 \log 2x}{2} - \frac{x^2}{4} + C$$
Where C is an arbitrary constant.

Question 6:

Integrate $x^2 \log x$

Solution 6:

Let $I = \int x^2 \log x dx$

Taking log x as first function and x^2 as second function and integrating by parts, we obtain

$$I = \log x \int x^2 dx - \int \left\{ \left(\frac{d}{dx} \log x \right) \int x^2 dx \right\} dx$$
$$= \log x . \left(\frac{x^3}{3} \right) - \int \frac{1}{x} . \frac{x^3}{3} dx$$
$$= \frac{x^3 \log x}{3} - \int \frac{x^2}{3} dx$$
$$= \frac{x^3 \log x}{3} - \frac{x^3}{9} + C$$
Where C is an arbitrary constant.

Question 7:

Integrate $x \sin^{-1} x$

Solution 7:

Let $I = \int x \sin^{-1} x dx$

Taking $\sin^{-1}$ x as first function and x as second function and integrating by parts, we obtain

$$I = \sin^{-1} x \int x \, dx - \int \left\{ \left(\frac{d}{dx} \sin^{-1} x \right) \int x \, dx \right\} dx$$
$$= \sin^{-1} x \left(\frac{x^2}{2} \right) - \int \frac{1}{\sqrt{1-x^2}} . \frac{x^2}{2} dx$$
$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \frac{-x^2}{\sqrt{1-x^2}} dx$$
$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \left\{ \frac{1-x^2}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} \right\} dx$$

$$=\frac{x^2\sin^{-1}x}{2}+\frac{1}{2}\int\left\{\sqrt{1-x^2}-\frac{1}{\sqrt{1-x^2}}\right\}dx$$

$$=\frac{x^2\sin^{-1}x}{2}+\frac{1}{2}\left\{\int\sqrt{1-x^2}dx-\int\frac{1}{\sqrt{1-x^2}}dx\right\}$$

$$=\frac{x^2\sin^{-1}x}{2}+\frac{1}{2}\left\{\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}x-\sin^{-1}x\right\}+C$$

$$=\frac{x^2\sin^{-1}x}{2}+\frac{x}{4}\sqrt{1-x^2}+\frac{1}{4}\sin^{-1}x-\frac{1}{2}\sin^{-1}x+C$$

$$=\frac{1}{4}\left(2x^2-1\right)\text{in}^{-1}x+\frac{x}{4}\sqrt{1-x^2}+C$$

Where C is an arbitrary constant.

Question 8:

Integrate $x\tan^{-1}x$

Solution 8:

Let $I=\int x\tan^{-1}x\,dx$

Taking $\tan^{-1}$ x as first function and x as second function and integrating by parts, we obtain

$$I=\tan^{-1}x\int x\,dx-\int\left\{\left(\frac{d}{dx}\tan^{-1}x\right)\int x\,dx\right\}dx$$

$$=\tan^{-1}x\left(\frac{x^2}{2}\right)-\int\frac{1}{1+x^2}\cdot\frac{x^2}{2}dx$$

$$=\frac{x^2\tan^{-1}x}{2}-\frac{1}{2}\int\frac{x^2}{1+x^2}dx$$

$$=\frac{x^2\tan^{-1}x}{2}-\frac{1}{2}\int\left(\frac{x^2+1}{1+x^2}-\frac{1}{1+x^2}\right)dx$$

$$=\frac{x^2\tan^{-1}x}{2}-\frac{1}{2}\int\left(1-\frac{1}{1+x^2}\right)dx$$

$$=\frac{x^2\tan^{-1}x}{2}-\frac{1}{2}\left(x-\tan^{-1}x\right)+C$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{x}{2}+\frac{1}{2}\tan^{-1}x+C$$

Where C is an arbitrary constant.

Question 9:

Integrate $x\cos^{-1}x$

Solution 9:

Let $I=\int x\cos^{-1}xdx$

Taking $\cos^{-1}$ x as first function and x as second function and integrating by parts, we obtain

$$I=\cos^{-1}x\int xdx-\int\left\{\left(\frac{d}{dx}\cos^{-1}x\right)\int xdx\right\}dx$$

$$=\cos^{-1}x\frac{x^2}{2}-\int\frac{-1}{\sqrt{1-x^2}}.\frac{x^2}{2}dx$$

$$=\frac{x^2\cos^{-1}x}{2}-\frac{1}{2}\int\frac{1-x^2-1}{\sqrt{1-x^2}}dx$$

$$=\frac{x^2\cos^{-1}x}{2}-\frac{1}{2}\int\left\{\sqrt{1-x^2}+\left(\frac{-1}{\sqrt{1-x^2}}\right)\right\}dx$$

$$=\frac{x^2\cos^{-1}x}{2}-\frac{1}{2}\int\sqrt{1-x^2}dx-\frac{1}{2}\int\left(\frac{-1}{\sqrt{1-x^2}}\right)dx$$

$$I=\frac{x\cos^{-1}x}{2}-\frac{1}{2}\left(\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}x\right)-\frac{1}{2}\cos^{-1}x$$

Where C is an arbitrary constant.

Question 10:

Integrate $\left(\sin^{-1}x\right)^2$

Solution 10:

Let $I=\int\left(\sin^{-1}x\right)^2.1dx$

Taking $(\sin^{-1}x)^2$ as first function and 1 as second function and integrating by parts, we obtain

$$I=\int\left(\sin^{-1}x\right).\int 1dx-\int\left\{\frac{d}{dx}\left(\sin^{-1}x\right)^2.\int 1.dx\right\}dx$$

$$=\left(\sin^{-1}x\right)^2.x-\int\frac{2\sin^{-1}x}{\sqrt{1-x^2}}.xdx$$

$$=x\left(\sin^{-1}x\right)^2+\int\sin^{-1}x.\left(\frac{-2x}{\sqrt{1-x^2}}\right)dx$$

$$=x\left(\sin^{-1}x\right)^2+\left[\sin^{-1}x\int\frac{-2x}{\sqrt{1-x^2}}dx-\int\left\{\left(\frac{d}{dx}\sin^{-1}x\right)\int\frac{-2x}{\sqrt{1-x^2}}dx\right\}dx\right]$$

$$=x\left(\sin^{-1}x\right)^2+\left[\sin^{-1}x.2\sqrt{1-x^2}-\int\frac{1}{\sqrt{1-x^2}}.2\sqrt{1-x^2}dx\right]$$

$$=x\left(\sin^{-1}x\right)^2+2\sqrt{1-x^2}\sin^{-1}x-\int 2dx$$

$$=x\left(\sin^{-1}x\right)^2+2\sqrt{1-x^2}\sin^{-1}x-2x+C$$

Where C is an arbitrary constant.

Question 11:

Integrate $\dfrac{x\cos^{-1}x}{\sqrt{1-x^2}}$

Solution 11:

Let $I=\int\dfrac{x\cos^{-1}x}{\sqrt{1-x^2}}dx$

$I=\dfrac{-1}{2}\int\dfrac{-2x}{\sqrt{1-x^2}}.\cos^{-1}xdx$

Taking $\cos^{-1}$ x as first function and $\left(\dfrac{-2x}{\sqrt{1-x^2}}\right)$ as second function and integrating by parts, we obtain

$$I=\frac{-1}{2}\left[\cos^{-1}x\int\frac{-2x}{\sqrt{1-x^2}}dx-\int\left\{\left(\frac{d}{dx}\cos^{-1}x\right)\int\frac{-2x}{\sqrt{1-x^2}}dx\right\}dx\right]$$

$$=\frac{-1}{2}\left[\cos^{-1}x.2\sqrt{1-x^2}-\int\frac{-1}{\sqrt{1-x^2}}.2\sqrt{1-x^2}dx\right]$$

$$=\frac{-1}{2}\left[2\sqrt{1-x^2}\cos^{-1}x+\int 2dx\right]$$

$$=\frac{-1}{2}\left[2\sqrt{1-x^2}\cos^{-1}x+2x\right]+C$$

$$=-\left[\sqrt{1-x^2}\cos^{-1}x+x\right]+C$$

Where C is an arbitrary constant.

Question 12:

Integrate $x\sec^2 x$

Solution 12:

Let $I=\int x\sec^2 xdx$

Taking x as first function and $\sec^2$ x as second function and integrating by parts, we obtain

$$I=x\int\sec^2 xdx-\int\left\{\left\{\frac{d}{dx}x\right\}\int\sec^2 xdx\right\}dx$$

$$=x\tan x-\int 1.\tan xdx$$

$$=x\tan x+\log|\cos x|+C$$

Where C is an arbitrary constant.

Question 13:

Integrate $\tan^{-1} x$

Solution 13:

Let $I = \int 1.\tan^{-1} x dx$

Taking $\tan^{-1}$ x as first function and 1 as second function and integrating by parts, we obtain

$$I = \tan^{-1} x \int 1 dx - \int \left\{ \left(\frac{d}{dx} \tan^{-1} x \right) \int 1.dx \right\} dx$$

$$= \tan^{-1} x.x - \int \frac{1}{1+x^2} .xdx$$

$$= x\tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2} dx$$

$$= x\tan^{-1} x - \frac{1}{2} \log\left|1+x^2\right| + C$$

$$= x\tan^{-1} x - \frac{1}{2} \log\left(1+x^2\right) + C$$

Where C is an arbitrary constant.

Question 14:

Integrate $x(\log x)^2 dx$

Solution 14:

$$I = \int x(\log x)^2 dx$$

Taking $(\log x)^2$ as first function and 1 as second function and integrating by parts, we obtain

$$I = (\log)^2 \int xdx - \int \left[\left\{ \left(\frac{d}{dx} \log x \right)^2 \right\} \int xdx \right] dx$$

$$= \frac{x^2}{2} (\log x)^2 - \left[\int 2\log x . \frac{1}{x} . \frac{x^2}{2} dx \right]$$

$$= \frac{x^2}{2} (\log x)^2 - \int x \log xdx$$

Again integrating by parts, we obtain

$$I = \frac{x^2}{2} (\log x)^2 - \left[\log x \int xdx - \int \left\{ \left(\frac{d}{dx} \log x \right) \int xdx \right\} dx \right]$$

$$= \frac{x^2}{2} (\log x)^2 - \left[\frac{x^2}{2} \log x - \int \frac{1}{x} \cdot \frac{x^2}{2} dx \right]$$

$$= \frac{x^2}{2} (\log x)^2 - \frac{x^2}{2} \log x + \frac{1}{2} \int xdx$$

$$=\frac{x^2}{2}(\log x)^2-\frac{x^2}{2}\log x+\frac{x^2}{4}+C$$

Where C is an arbitrary constant.

Question 15:

Integrate $\left(x^2+1\right)\log x$

Solution 15:

Let $I=\int\left(x^2+1\right)\log x dx=\int x^2\log x dx+\int \log x dx$

Let I = I_1 + I_2 … (1)

Where, $I_1=\int x^2\log x dx$ and $I_2=\int \log x dx$

$$I_1=\int x^2\log x dx$$

Taking log x as first function and x^2 as second function and integrating by parts, we obtain

$$I_1=\log x\int x^2\,dx-\int\left\{\left(\frac{d}{dx}\log x\right)\int x^2\,dx\right\}dx$$

$$=\log x.\frac{x^3}{3}-\int\frac{1}{x}.\frac{x^3}{3}dx$$

$$=\frac{x^3}{3}\log x-\frac{1}{3}\left(\int x^2dx\right)$$

$$=\frac{x^3}{3}\log x-\frac{x^3}{9}+C_1 \qquad ...(2)$$

$$I_2=\int \log x dx$$

Taking log x as first function and 1 as second function and integrating by parts, we obtain

$$I_2=\log x\int 1.dx-\int\left\{\left(\frac{d}{dx}\log x\right)\int 1.dx\right\}$$

$$=\log x.x-\int\frac{1}{x}.xdx$$

$$=x\log x-\int 1dx$$

$$=x\log x-x+C_2 \qquad ...(3)$$

Using equations (2) and (3) in (1), we obtain

$$I=\frac{x^3}{3}\log x-\frac{x^3}{9}+C_1+x\log x-x+C_2$$

$$=\frac{x^3}{3}\log x-\frac{x^3}{9}+x\log x-x+\left(C_1+C_2\right)$$

$$=\left(\frac{x^3}{3}+x\right)\log x-\frac{x^3}{9}-x+C$$

Where C is an arbitrary constant.

Question 16:

Integrate $e^x(\sin x+\cos x)$

Solution 16:

Let $I=\int e^x(\sin x+\cos x)dx$

Let f (x) = sin x

$f'(x)=\cos x$

$I=\int e^x\{f(x)+f'(x)\}dx$

It is known that, $\int e^x\{f(x)+f'(x)\}dx=e^x f(x)+C$

$\therefore I=e^x\sin x+C$

Where C is an arbitrary constant.

Question 17:

Integrate $\frac{xe^x}{(1+x)^2}$

Solution 17:

Let $I=\int\frac{xe^x}{(1+x)^2}dx=\int e^x\left\{\frac{x}{(1+x)^2}\right\}dx$

$=\int e^x\left\{\frac{1+x-1}{(1+x)^2}\right\}dx$

$=\int e^x\left\{\frac{1}{1+x}-\frac{1}{(1+x)^2}\right\}dx$

Let $f(x)=\frac{1}{1+x}$ $\quad f'(x)=\frac{-1}{(1+x)^2}$

$\Rightarrow\int\frac{xe^x}{(1+x)^2}dx=\int e^x\{f(x)+f'(x)\}dx$

It is known that, $\int e^x\{f(x)+f'(x)\}dx=e^x f(x)+C$

$\therefore\int\frac{xe^x}{(1+x)^2}dx=\frac{e^x}{1+x}+C$

Where C is an arbitrary constant.

Question 18:

Integrate $e^x\left(\frac{1+\sin x}{1+\cos x}\right)$

Solution 18:

$$e^x\left(\frac{1+\sin x}{1+\cos x}\right)$$

$$=e^x\left(\frac{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right)$$

$$=\frac{e^x\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2}{2\cos^2\frac{x}{2}}$$

$$=\frac{1}{2}e^x.\left(\frac{\sin\frac{x}{2}+\cos\frac{x}{2}}{\cos\frac{x}{2}}\right)^2$$

$$=\frac{1}{2}e^x\left[\tan\frac{x}{2}+1\right]^2$$

$$=\frac{1}{2}e^x\left(1+\tan\frac{x}{2}\right)^2$$

$$=\frac{1}{2}e^x\left[1+\tan^2\frac{x}{2}+2\tan\frac{x}{2}\right]$$

$$=\frac{1}{2}e^x\left[\sec^2\frac{x}{2}+2\tan\frac{x}{2}\right]$$

$$\frac{e^x(1+\sin x)dx}{(1+\cos x)}=e^x\left[\frac{1}{2}\sec^2\frac{x}{2}+\tan\frac{x}{2}\right] \qquad ...(1)$$

Let $\tan\frac{x}{2}=f(x)$ *so* $f'(x)=\frac{1}{2}\sec^2\frac{x}{2}$

It is known that, $\int \mathrm{e}^x\{f(x)+f'(x)\}dx=e^x f(x)+C$

From equation (1), we obtain

$$\int\frac{e^x(1+\sin x)}{(1+\cos x)}dx=e^x\tan\frac{x}{2}+C$$

Where C is an arbitrary constant.

Question 19:

Integrate $e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)$

Solution 19:

Let $I = \int e^x \left[\frac{1}{x} - \frac{1}{x^2} \right] dx$

Also, let $\frac{1}{x} = f(x) \qquad f'(x) = \frac{-1}{x^2}$

It is known that, $\int e^x \{ f(x) + f'(x) \} dx = e^x f(x) + C$

$\therefore I = \frac{e^x}{x} + C$

Where C is an arbitrary constant.

Question 20:

Integrate $\frac{(x-3)e^x}{(x-1)^3}$

Solution 20:

$$\int e^x \left\{ \frac{x-3}{(x-1)^3} \right\} dx = \int e^x \left\{ \frac{x-1-2}{(x-1)^3} \right\} dx$$

$$= \int e^x \left\{ \frac{1}{(x-1)^2} - \frac{2}{(x-1)^3} \right\} dx$$

Let $f(x) = \frac{1}{(x-1)^2} \qquad f'(x) = \frac{-2}{(x-1)^3}$

It is known that, $\int e^x \{ f(x) + f'(x) \} dx = e^x f(x) + C$

$$\therefore \int e^x \left\{ \frac{(x-3)}{(x-1)^2} \right\} dx = \frac{e^x}{(x-1)^2} + C$$

Where C is an arbitrary constant.

Question 21:

Integrate $e^{2x} \sin x$

Solution 21:

Let $I = \int e^{2x} \sin x dx \qquad ...(1)$

Integrating by parts, we obtain

$$I = \sin x \int e^{2x} dx - \int \left\{ \left(\frac{d}{dx} \sin x \right) \int e^{2x} dx \right\} dx$$

$$\Rightarrow I = \sin x . \frac{e^{2x}}{2} - \int \cos x . \frac{e^{2x}}{2} dx$$

$$\Rightarrow I = \frac{e^{2x}\sin x}{2} - \frac{1}{2}\int e^{2x}\cos x dx$$

Again integrating by parts, we obtain

$$I = \frac{e^{2x}\sin x}{2} - \frac{1}{2}\left[\cos x\int e^{2x}dx - \int\left\{\left(\frac{d}{dx}\cos x\right)\int e^{2x}dx\right\}dx\right]$$

$$\Rightarrow I = \frac{e^{2x}\sin x}{2} - \frac{1}{2}\left[\cos x.\frac{e^{2x}}{2} - \int(-\sin x)\frac{e^{2x}}{2}dx\right]$$

$$\Rightarrow I = \frac{e^{2x}\sin x}{2} - \frac{1}{2}\left[\frac{e^{2x}\cos x}{2} + \frac{1}{2}\int e^{2x}\sin x dx\right]$$

$$\Rightarrow I = \frac{e^{2x}\sin x}{2} - \frac{e^{2x}\cos x}{4} - \frac{1}{4}I \qquad \left[\text{From (1)}\right]$$

$$\Rightarrow I + \frac{1}{4}I = \frac{e^{2x}\sin x}{2} - \frac{e^{2x}\cos x}{4}$$

$$\Rightarrow \frac{5}{4}I = \frac{e^{2x}\sin x}{2} - \frac{e^{2x}\cos x}{4}$$

$$\Rightarrow I = \frac{4}{5}\left[\frac{e^{2x}\sin x}{2} - \frac{e^{2x}\cos x}{4}\right] + C$$

$$\Rightarrow I = \frac{e^{2x}}{5}\left[2\sin x - \cos x\right] + C$$

Where C is an arbitrary constant.

Question 22:

Integrate $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Solution 22:

Let $x = \tan\theta \qquad dx = \sec^2\theta d\theta$

$$\therefore \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta$$

$$\int\sin^{-1}\left(\frac{2x}{1+x^2}\right)dx = \int 2\theta.\sec^2\theta d\theta = 2\int\theta.\sec^2\theta d\theta$$

Integrating by parts, we obtain

$$2\left[\theta.\int\sec^2\theta d\theta - \int\left\{\left(\frac{d}{d\theta}\theta\right)\int\sec^2\theta d\theta\right\}d\theta\right]$$

$$= 2\left[\theta.\tan\theta - \int\tan\theta d\theta\right]$$

$$= 2\left[\theta\tan\theta + \log\left|\cos\theta\right|\right] + C$$

$$= 2\left[x\tan^{-1}x + \log\left|\frac{1}{\sqrt{1+x^2}}\right|\right] + C$$

$$= 2x\tan^{-1}x + 2\left[-\frac{1}{2}\log\left(1+x^2\right)\right] + C$$

$$= 2x\tan^{-1}x - \log\left(1+x^2\right) + C$$

Where C is an arbitrary constant.

Question 23:

$\int x^2 e^{x^3} dx$ equals

(A) $\frac{1}{3}e^{x^3} + C$ (B) $\frac{1}{3}e^{x^2} + C$

(C) $\frac{1}{2}e^{x^3} + C$ (D) $\frac{1}{3}e^{x^2} + C$

Solution 23:

Let $I = \int x^2 e^{x^3} dx$

Also, let x^3 = t so $3x^2 dx = dt$

$$\Rightarrow I = \frac{1}{3}\int e^t dt$$

$$= \frac{1}{3}\left(e^t\right) + C$$

$$= \frac{1}{3}e^{x^3} + C$$

Hence, the correct Answer is A.

Question 24:

$\int e^x \sec x\left(1+\tan x\right)dx$ equals

(A) $e^x \cos x + C$ (B) $e^x \sec x + C$

(C) $e^x \sin x + C$ (D) $e^x \tan x + C$

Solution 24:

$\int e^x \sec x\left(1+\tan x\right)dx$

Let $I = \int e^x \sec x\left(1+\tan x\right)dx = \int e^x\left(\sec x + \sec x\tan x\right)dx$

Also, let $\sec x = f(x)$ $\sec x\tan x = f'(x)$

It is known that, $\int e^x\left\{f(x) + f'(x)\right\}dx = e^x f(x) + C$

$\therefore I = e^x \sec x + C$

Hence, the correct Answer is B.

Exercise 7.7

Question 1:

Integrate $\sqrt{4-x^2}$

Solution 1:

Let $I=\int\sqrt{4-x^2}dx=\int\sqrt{(2)^2-(x)^2}dx$

It is known that,

$\sqrt{a^2-x^2}dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$

$\therefore I=\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\frac{x}{2}+C$

$=\frac{x}{2}\sqrt{4-x^2}+2\sin^{-1}\frac{x}{2}+C$

Where C is an arbitrary constant.

Question 2:

Integrate $\sqrt{1-4x^2}$

Solution 2:

Let $I=\int\sqrt{1-4x^2}dx=\int\sqrt{(1)^2-(2x)^2}dx$

Let $2x=t\Rightarrow 2dx=dt$

$\therefore I=\frac{1}{2}\int\sqrt{(1)^2-(t)^2}$

It is known that,

$\sqrt{a^2-x^2}dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$

$\Rightarrow I=\frac{1}{2}\left[\frac{t}{2}\sqrt{1-t^2}+\frac{1}{2}\sin^{-1}t\right]+C$

$=\frac{t}{4}\sqrt{1-t^2}+\frac{1}{4}\sin^{-1}t+C$

$=\frac{2x}{4}\sqrt{1-4x^2}+\frac{1}{4}\sin^{-1}2x+C$

$=\frac{x}{2}\sqrt{1-4x^2}+\frac{1}{4}\sin^{-1}2x+C$

Where C is an arbitrary constant.

Question 3:

Integrate $\sqrt{x^2+4x+6}$

Solution 3:

Let $I=\int\sqrt{x^2+4x+6}dx$

$=\int\sqrt{x^2+4x+4+2}dx$

$=\int\sqrt{\left(x^2+4x+4\right)+2}dx$

$=\int\sqrt{(x+2)^2+\left(\sqrt{2}\right)^2}\,dx$

It is known that,

$\sqrt{x^2+a^2}dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log|x+\sqrt{x^2+a^2}|+C$

$\therefore I=\frac{(x+2)}{2}\sqrt{x^2+4x+6}+\frac{2}{2}\log\left|(x+2)+\sqrt{x^2+4x+6}\right|+C$

$=\frac{(x+2)}{2}\sqrt{x^2+4x+6}+\log\left|(x+2)+\sqrt{x^2+4x+6}\right|+C$

Where C is an arbitrary constant.

Question 4:

Integrate $\sqrt{x^2+4x+1}$

Solution 4:

Let $I=\int\sqrt{x^2+4x+1}dx$

$=\int\sqrt{\left(x^2+4x+4\right)-3}dx$

$=\int\sqrt{(x+2)^2-\left(\sqrt{3}\right)^2}\,dx$

It is known that,

$\sqrt{x^2-a^2}dx=\frac{x}{2}\sqrt{x^2-a^2}+\frac{a^2}{2}\log|x+\sqrt{x^2-a^2}|+C$

$\therefore I=\frac{(x+2)}{2}\sqrt{x^2+4x+1}-\frac{3}{2}\log\left|(x-2)+\sqrt{x^2+4x+1}\right|+C$

Where C is an arbitrary constant.

Question 5:

Integrate $\sqrt{1-4x-x^2}$

Solution 5:

Let $I=\int\sqrt{1-4x-x^2}\,dx$

$=\int\sqrt{1-\left(x^2+4x+4-4\right)}dx$

$=\int\sqrt{1+4-(x+2)^2}\,dx$

$=\int\sqrt{\left(\sqrt{5}\right)^2-(x+2)^2}\,dx$

It is known that,

$\sqrt{a^2-x^2}dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$

$\therefore I=\frac{(x+2)}{2}\sqrt{1-4x-x^2}+\frac{5}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right)+C$

Where C is an arbitrary constant.

Question 6:

Integrate $\sqrt{x^2+4x-5}$

Solution 6:

Let $I=\int\sqrt{x^2+4x-5}dx$

$=\int\sqrt{\left(x^2+4x+4\right)-9}dx$

$=\int\sqrt{(x+2)^2-(3)^2}\,dx$

It is known that, $\int\sqrt{x^2-a^2}\ dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$

$\therefore I=\frac{(x+2)}{2}\sqrt{x^2+4x-5}-\frac{9}{2}\log\left|(x+2)+\sqrt{x^2+4x-5}\right|+C$

Where C is an arbitrary constant.

Question 7:

Integrate $\sqrt{1+3x-x^2}$

Solution 7:

Let $I=\int\sqrt{1+3x-x^2}dx$

$=\int\sqrt{1-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)}dx$

$=\int\sqrt{\left(1+\frac{9}{4}\right)-\left(x-\frac{3}{2}\right)^2}\ dx$

$$=\int\sqrt{\left(\frac{\sqrt{13}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}\,dx$$

It is known that,

$$\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$$

$$\therefore I=\frac{x-\frac{3}{2}}{2}\sqrt{1+3x-x^2}+\frac{13}{4\times 2}\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{13}}{2}}\right)+C$$

$$=\frac{2x-3}{4}\sqrt{1+3x-x^2}+\frac{13}{8}\sin^{-1}\left(\frac{2x-3}{\sqrt{13}}\right)+C$$

Where C is an arbitrary constant.

Question 8:

Integrate $\sqrt{x^2+3x}$

Solution 8:

Let $I=\int\sqrt{x^2+3x}\,dx$

$$=\int\sqrt{x^2+3x+\frac{9}{4}-\frac{9}{4}}dx$$

$$=\int\sqrt{\left(x+\frac{3}{4}\right)^2-\left(\frac{3}{2}\right)^2}\,dx$$

It is known that,

$$\sqrt{x^2-a^2}dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$$

$$\therefore I=\frac{\left(x+\frac{3}{2}\right)}{2}\sqrt{x^2+3x}-\frac{\frac{9}{4}}{2}\log\left|\left(x+\frac{3}{2}\right)+\sqrt{x^2+3x}\right|+C$$

$$=\frac{(2x+3)}{4}\sqrt{x^2+3x}-\frac{9}{8}\log\left|\left(x+\frac{3}{2}\right)+\sqrt{x^2+3x}\right|+C$$

Where C is an arbitrary constant.

Question 9:

Integrate $\sqrt{1+\frac{x^2}{9}}$

Solution 9:

Let $I=\int\sqrt{1+\frac{x^2}{9}}dx=\frac{1}{3}\int\sqrt{9+x^2}dx=\frac{1}{3}\int\sqrt{(3)^2+x^2}dx$

It is known that,

$$\sqrt{x^2+a^2}dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+C$$

$$\therefore I=\frac{1}{3}\left[\frac{x}{2}\sqrt{x^2+9}+\frac{9}{2}\log\left|x+\sqrt{x^2+9}\right|\right]+C$$

$$=\frac{x}{6}\sqrt{x^2+9}+\frac{3}{2}\log\left|x+\sqrt{x^2+9}\right|+C$$

Where C is an arbitrary constant.

Question 10:

$\int\sqrt{1+x^2}$ is equal to

A. $\frac{x}{2}\sqrt{1+x^2}+\frac{1}{2}\log\left|x+\sqrt{1+x^2}\right|+C$

B. $\frac{2}{3}\left(1+x^2\right)^{\frac{2}{3}}+C$

C. $\frac{2}{3}x\left(1+x^2\right)^{\frac{2}{3}}+C$

D. $\frac{x^3}{2}\sqrt{1+x^2}+\frac{1}{2}x^2\log\left|x+\sqrt{1+x^2}\right|+C$

Solution 10:

It is known that,

$$\sqrt{x^2+a^2}dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log|x+\sqrt{x^2+a^2}|+C$$

$$\therefore\int\sqrt{1+x^2}dx=\frac{x}{2}\sqrt{1+x^2}+\frac{1}{2}\log\left|x+\sqrt{1+x^2}\right|+C$$

Hence, the correct Answer is A.

Question 11:

$\int\sqrt{x^2-8x+7}$dx is equal to

A. $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}+9\log\left|x-4+\sqrt{x^2-8x+7}\right|+C$

B. $\frac{1}{2}(x+4)\sqrt{x^2-8x+7}+9\log\left|x+4+\sqrt{x^2-8x+7}\right|+C$

C. $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}-3\sqrt{2}\log\left|x-4+\sqrt{x^2-8x+7}\right|+C$

D. $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}-\frac{9}{2}\log\left|x-4+\sqrt{x^2-8x+7}\right|+C$

Solution 11:

Let $I=\int\sqrt{x^2-8x+7}dx$

$=\int\sqrt{\left(x^2-8x+16\right)-9}dx$

$=\int\sqrt{(x-4)^2-(3)^2}dx$

It is known that, $\sqrt{x^2-a^2}dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$

$\therefore I=\frac{(x-4)}{2}\sqrt{x^2-8x+7}-\frac{9}{2}\log\left|(x-4)+\int\sqrt{x^2-8x+7}\right|+C$

Hence, the correct Answer is D.

Exercise 7.8

Question 1:

$\int_a^b xdx$

Solution 1:

It is known that,

$\int_a^b f(x)dx=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+\ldots+f(a+(n-1)h)\right]$ where $h=\frac{b-a}{n}$

$Here, a=a, b=b,$ and $f(x)=x$

$\therefore\int_a^b xdx=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[a+(a+h)\ldots(a+2h)\ldots a+(n-1)h\right]$

$=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[\left(a+\underbrace{a}_{ntimes}+a+\ldots+a\right)+\left(h+2h+3h+\ldots+(n-1)h\right)\right]$

$=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[na+h\left(1+2+3+\ldots+(n-1)\right)\right]$

$=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[na+h\left\{\frac{(n-1)(n)}{2}\right\}\right]$

$=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[na+\frac{n(n-1)h}{2}\right]$

$$=(b-a)\lim_{n\to\infty}\frac{n}{n}\left[a+\frac{(n-1)h}{2}\right]$$

$$=(b-a)\lim_{n\to\infty}\left[a+\frac{(n-1)h}{2}\right]$$

$$=(b-a)\lim_{n\to\infty}\left[a+\frac{(n-1)(b-a)}{2n}\right]$$

$$=(b-a)\lim_{n\to\infty}\left[a+\frac{\left(1-\frac{1}{n}\right)(b-a)}{2}\right]$$

$$=(b-a)\left[a+\frac{(b-a)}{2}\right]$$

$$=(b-a)\left[\frac{2a+b-a}{2}\right]$$

$$=\frac{(b-a)(b+a)}{2}$$

$$=\frac{1}{2}\left(b^2-a^2\right)$$

Question 2:

$\int_0^b (x+1)\,dx$

Solution 2:

Let I = $\int_0^b (x+1)\,dx$

It is known that,

$$\int_a^b f(x)dx=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+\ldots+f\left(a+(n-1)h\right)\right],\text{ where } h=\frac{b-a}{n}$$

$Here, a=0, b=5,$ and $f(x)=(x+1)$

$$\Rightarrow h=\frac{5-0}{n}=\frac{5}{n}$$

$$\therefore \int_0^5 (x+1)dx=(5-0)\lim_{n\to\infty}\frac{1}{n}\left[f(0)+f\left(\frac{5}{n}\right)+\ldots f\left((n-1)\frac{5}{n}\right)\right]$$

$$=5\lim_{n\to\infty}\frac{1}{n}\left[1+\left(\frac{5}{n}+1\right)+\ldots\left\{1+\left(\frac{5(n-1)}{n}\right)\right\}\right]$$

$$=5\lim_{n\to\infty}\frac{1}{n}\left[\left(\underbrace{1+1+1\ldots1}_{n\text{ times}}\right)+\left[\frac{5}{n}+2.\frac{5}{n}+3.\frac{5}{n}+\ldots(n-1)\frac{5}{n}\right]\right]$$

$$=5\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{5}{n}\{1+2+3...(n-1)\}\right]$$

$$=5\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{5}{n}.\frac{(n-1)n}{2}\right]$$

$$=5\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{5(n-1)}{2}\right]$$

$$=5\lim_{n\to\infty}\frac{1}{n}\left[1+\frac{5}{2}\left(1-\frac{1}{n}\right)\right]$$

$$=5\left[1+\frac{5}{2}\right]$$

$$=5\left[\frac{7}{2}\right]$$

$$=\frac{35}{2}$$

Question 3:

$$\int_2^3 x^2dx$$

Solution 3:

It is known that,

$$\int_a^b f(x)dx=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+f(a+2h)...+f\{a+(n-1)h\}\right],\text{ where } h=\frac{b-a}{n}$$

Here, $a=2, b=3,$ and $f(x)=x^2$

$$\Rightarrow h=\frac{3-2}{n}=\frac{1}{n}$$

$$\therefore \int_2^3 x^2dx=(3-2)\lim_{n\to\infty}\frac{1}{n}\left[f(2)+f\left(2+\frac{1}{n}\right)+f\left(2+\frac{2}{n}\right)...f\left\{2+(n-1)\frac{1}{n}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[(2)^2+\left(2+\frac{1}{n}\right)^2+\left(2+\frac{2}{n}\right)^2+...\left(2+\frac{(n-1)}{n}\right)^2\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[2^2+\left\{2^2+\left(\frac{1}{n}\right)^2+2.2.\frac{1}{n}\right\}+...+\left\{(2)^2+\frac{(n-1)^2}{n^2}+2.2.\frac{(n-1)}{n}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[\left(2^2+\underset{n\ times}{...}+2^2\right)+\left\{\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+...+\left(\frac{n-1}{n}\right)^2\right\}+2.2.\left\{\frac{1}{n}+\frac{2}{n}+\frac{3}{n}+...+\frac{(n-1)}{n}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[4n+\frac{1}{n^2}\left\{1^2+2^2+3^2...+(n-1)^2\right\}+\frac{4}{n}\{1+2+...+(n-1)\}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[4n+\frac{1}{n^2}\left\{\frac{n(n-1)(2n-1)}{6}\right\}+\frac{4}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[4n+\frac{n\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)}{6}+\frac{4n-4}{2}\right]$$

$$= \lim_{n\to\infty}\left[4+\frac{1}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+2-\frac{2}{n}\right]$$

$$= 4+\frac{2}{6}+2$$

$$= \frac{19}{3}$$

Question 4:

$$\int_1^4 (x^2-x)\,dx$$

Solution 4:

Let $I=\int_1^4 (x^2-x)\,dx$

$$= \int_1^4 x^2dx-\int_1^4 xdx$$

Let $I=I_1-I_2$, where $I_1=\int_1^4 x^2dx$ and $I_2=\int_1^4 xdx$ ...(1)

It is known that,

$$\int_a^b f(x)dx=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+f(a+(n-1)h)\right],\text{ where } h=\frac{b-a}{n}$$

For, $I_1=\int_1^4 x^2dx$,

a = 1, b = 4, and f (x) = x^2

$$\therefore h=\frac{4-1}{n}=\frac{3}{n}$$

$$I_1=\int_1^4 x^2dx=(4-1)\lim_{n\to\infty}\frac{1}{n}\left[f(1)+f(1+h)+\ldots+f(1+(n-1)h)\right]$$

$$=3\lim_{n\to\infty}\frac{1}{n}\left[1^2+\left(1+\frac{3}{n}\right)^2+\left(1+2.\frac{3}{n}\right)^2+\ldots\left(1+\frac{(n-1)3}{n}\right)^2\right]$$

$$=3\lim_{n\to\infty}\frac{1}{n}\left[1^2+\left\{1^2+\left(\frac{3}{n}\right)^2+2.\frac{3}{n}\right\}+\ldots+\left\{1^2+\left(\frac{(n-1)3}{n}\right)^2+\frac{2.(n-1).3}{2}\right\}\right]$$

$$=3\lim_{n\to\infty}\frac{1}{n}\left[\left(1^2+\underset{n\,times}{\ldots}+1^2\right)+\left(\frac{3}{n}\right)^2\left\{1^2+2^2+\ldots+(n-1)^2\right\}+2.\frac{3}{n}\left\{1+2+\ldots+(n-1)\right\}\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{9}{n^2}\left\{\frac{(n-1)(n)(2n-1)}{6}\right\}+\frac{6}{n}\left\{\frac{(n-1)(n)}{2}\right\}\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{9n}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+\frac{6n-6}{2}\right]$$

$$= 3\lim_{n\to\infty}\left[1+\frac{9}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+3-\frac{3}{n}\right]$$

$$= 3[1+3+3]$$

$$= 3[7]$$

$$I_1 = 21 \qquad \ldots(2)$$

For $I_2 = \int_1^4 x\,dx,$

$a = 1, b = 4,$ and $f(x) = x$

$$\Rightarrow h = \frac{4-1}{n} = \frac{3}{n}$$

$$\therefore I_2 = (4-1)\lim_{n\to\infty}\frac{1}{n}\left[f(1)+f(1+h)+\ldots f(a+(n-1)h)\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[1+(1+h)+\ldots+(1+(n-1)h)\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[1+\left(1+\frac{3}{n}\right)+\ldots+\left\{1+(n-1)\frac{3}{n}\right\}\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[\left(\underbrace{1+1+\ldots+1}_{n\ \text{times}}\right)+\frac{3}{n}\left(1+2+\ldots+(n-1)\right)\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{3}{n}\left\{\frac{(n-1)n}{2}\right\}\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[1+\frac{3}{2}\left(1-\frac{1}{n}\right)\right]$$

$$= 3\left[1+\frac{3}{2}\right]$$

$$= 3\left[\frac{5}{2}\right]$$

$$I_2 = \frac{15}{2} \qquad \ldots(3)$$

From equations (2) and (3), we obtain

$$I = I_1 - I_2 = 21 - \frac{15}{2} = \frac{27}{2}$$

Question 5:

$$\int_{-1}^{1} e^x\,dx$$

Solution 5:

Let $I=\int_{-1}^{1} e^x dx \qquad ...(1)$

It is known that,

$$\int_a^b f(x)dx=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+...+f(a+(n-1)h)\right] \text{ where } h=\frac{b-a}{n}$$

$Here, a=-1, b=1, \text{ and } f(x)=e^x$

$$\therefore h=\frac{1+1}{n}=\frac{2}{n}$$

$$\therefore I=(1+1)\lim_{n\to\infty}\frac{1}{n}\left[f(-1)+f\left(-1+\frac{2}{n}\right)+f\left(-1+2.\frac{2}{n}\right)+...+f\left(-1+\frac{(n-1)2}{n}\right)\right]$$

$$=2\lim_{n\to\infty}\frac{1}{n}\left[e^{-1}+e^{\left(-1+\frac{2}{n}\right)}+e^{\left(-1+2.\frac{2}{n}\right)}+...e^{\left(-1+(n-1)\frac{2}{n}\right)}\right]$$

$$=2\lim_{n\to\infty}\frac{1}{n}\left[e^{-1}\left\{1+e^{\frac{2}{n}}+e^{\frac{4}{n}}+e^{\frac{6}{n}}+e^{(n-1)\frac{2}{n}}\right\}\right]$$

$$=2\lim_{n\to\infty}\frac{e^{-1}}{n}\left[\frac{e^{\frac{2n}{n}}-1}{e^{\frac{2}{n}}-1}\right]$$

$$=e^{-1}\times 2\lim_{n\to\infty}\frac{1}{n}\left[\frac{e^2-1}{e^{\frac{2}{n}}-1}\right]$$

$$=\frac{e^{-1}\times 2\left(e^2-1\right)}{\lim\limits_{\frac{2}{n}\to 0}\left(\dfrac{e^{\frac{2}{n}}-1}{\frac{2}{n}}\right)\times 2}$$

$$=e^{-1}\left[\frac{2\left(e^2-1\right)}{2}\right] \qquad \left[\lim_{h\to 0}\left(\frac{e^h-1}{h}\right)=1\right]$$

$$=\frac{e^2-1}{e}$$

$$=\left(e-\frac{1}{e}\right)$$

Question 6:

$$\int_0^4\left(x+e^{2x}\right)dx$$

Solution 6:

It is known that,

$$\int_a^b f(x)dx=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+\cdots+f\left(a+(n-1)h\right)\right] \text{ where } h=\frac{b-a}{n}$$

$Here, a=0, b=4,$ and $f(x)=x+e^{2x}$

$$\therefore h=\frac{4-0}{n}=\frac{4}{n}$$

$$\Rightarrow \int_0^4\left(x+e^{2x}\right)dx=(4-0)\lim_{n\to\infty}\frac{1}{n}\left[f(0)+f(h)+f(2h)+\ldots+f\left((n-1)h\right)\right]$$

$$=4\lim_{n\to\infty}\frac{1}{n}\left[\left(0+e^0\right)+\left(h+e^{2h}\right)+\left(2h+e^{2.2h}\right)+\ldots+\left\{(n-1)h+e^{2(n-1)h}\right\}\right]$$

$$=4\lim_{n\to\infty}\frac{1}{n}\left[1+\left(h+e^{2h}\right)+\left(2h+e^{4h}\right)+\ldots+\left\{(n-1)h+e^{2(n-1)h}\right\}\right]$$

$$=4\lim_{n\to\infty}\frac{1}{n}\left[\left\{h+2h+3h+\ldots+(n-1)h\right\}+\left(1+e^{2h}+e^{4h}+\ldots+e^{2(n-1)h}\right)\right]$$

$$=4\lim_{n\to\infty}\frac{1}{n}\left[h\left\{1+2+\ldots(n-1)\right\}+\left(\frac{e^{2hn}-1}{e^{2h}-1}\right)\right]$$

$$=4\lim_{n\to\infty}\frac{1}{n}\left[\frac{\left(h(n-1)n\right)}{2}+\left(\frac{e^{2hn}-1}{e^{2h}-1}\right)\right]$$

$$=4\lim_{n\to\infty}\frac{1}{n}\left[\frac{4}{n}.\frac{(n-1)n}{2}+\left(\frac{e^8-1}{e^{\frac{8}{n}}-1}\right)\right]$$

$$=4(2)+4\lim_{n\to\infty}\frac{\left(e^8-1\right)}{\left(\dfrac{e^{\frac{8}{n}}-1}{\frac{8}{n}}\right)8}$$

$$=8+\frac{4.\left(e^8-1\right)}{8} \qquad \left(\lim_{x\to 0}\frac{e^x-1}{x}=1\right)$$

$$=8+\frac{e^8-1}{2}$$

$$=\frac{15+e^8}{2}$$

Exercise 7.9

Question 1:

$$\int_{-1}^{1}(x+1)dx$$

Solution 1:

Let $I=\int_{-1}^{1}(x+1)\,dx$

$\int(x+1)\,dx=\frac{x^2}{2}+x=F(x)$

By second fundamental theorem of calculus, we obtain

$I=F(1)-F(-1)$

$=\left(\frac{1}{2}+1\right)-\left(\frac{1}{2}-1\right)$

$=\frac{1}{2}+1-\frac{1}{2}+1$

$=2$

Question 2:

$\int_{2}^{3}\frac{1}{x}dx$

Solution 2:

Let $I=\int_{2}^{3}\frac{1}{x}dx$

$\int\frac{1}{x}dx=\log|x|=F(x)$

By second fundamental theorem of calculus, we obtain

$I=F(3)-F(2)$

$=\log|3|-\log|2|=\log\frac{3}{2}$

Question 3:

$\int_{1}^{2}\left(4x^3-5x^2+6x+9\right)dx$

Solution 3:

Let $I=\int_{1}^{2}\left(4x^3-5x^2+6x+9\right)dx$

$\int\left(4x^3-5x^2+6x+9\right)dx=4\left(\frac{x^4}{4}\right)-5\left(\frac{x^3}{3}\right)+6\left(\frac{x^2}{2}\right)+9(x)$

$=x^4-\frac{5x^3}{3}+3x^2+9x=F(x)$

By second fundamental theorem of calculus, we obtain

I = F (2) – F (1)

$$I=\left\{2^4-\frac{5.(2)^3}{3}+3(2)^2+9(2)\right\}-\left\{(1)^4-\frac{5(1)^3}{3}+3(1)^2+9(1)\right\}$$

$$=\left(16-\frac{40}{3}+12+18\right)-\left(1-\frac{5}{3}+3+9\right)$$

$$=16-\frac{40}{3}+12+18-1+\frac{5}{3}-3-9$$

$$=33-\frac{35}{3}$$

$$=\frac{99-35}{3}$$

$$=\frac{64}{3}$$

Question 4:

$$\int_0^{\frac{x}{4}}\sin 2x dx$$

Solution 4:

Let $I=\int_0^{\frac{x}{4}}\sin 2x dx$

$$\int \sin 2x dx=\left(\frac{-\cos 2x}{2}\right)=F(x)$$

By second fundamental theorem of calculus, we obtain

$$I=F\left(\frac{\pi}{4}\right)-F(0)$$

$$=-\frac{1}{2}\left[\cos 2\left(\frac{\pi}{4}\right)-\cos 0\right]$$

$$=-\frac{1}{2}[0-1]$$

$$=\frac{1}{2}$$

Question 5:

$$\int_0^{\frac{\pi}{2}}\cos 2x dx$$

Solution 5:

Let $I=\int_0^{\frac{\pi}{2}}\cos 2x dx$

$$\int \cos 2x dx = \left(\frac{\sin 2x}{2}\right) = F(x)$$

By second fundamental theorem of calculus, we obtain

$$I = F\left(\frac{\pi}{2}\right) - F(0)$$

$$= \frac{1}{2}\left[\sin 2\left(\frac{\pi}{2}\right) - \sin 0\right]$$

$$= \frac{1}{2}[\sin \pi - \sin 0]$$

$$= \frac{1}{2}[0-0] = 0$$

Question 6:

$$\int_4^5 e^x dx$$

Solution 6:

Let $I = \int_4^5 e^x dx$

$$\int e^x dx = e^x = F(x)$$

By second fundamental theorem of calculus, we obtain

I = F (5) – F (4)

$$= e^5 - e^4$$

$$= e^4(e-1)$$

Question 7:

$$\int_0^{\frac{\pi}{4}} \tan x dx$$

Solution 7:

Let $I = \int_0^{\frac{\pi}{4}} \tan x dx$

$$\int \tan x dx = -\log|\cos x| = F(x)$$

By second fundamental theorem of calculus, we obtain

$$I = F\left(\frac{\pi}{4}\right) - F(0)$$

$$= -\log\left|\cos\frac{\pi}{4}\right| + \log|\cos 0|$$

$$= -\log\left|\frac{1}{\sqrt{2}}\right| + \log|1|$$

$$= -\log(2)^{-\frac{1}{2}}$$
$$= \frac{1}{2}\log 2$$

Question 8:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x\, dx$$

Solution 8:

Let $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x\, dx$

$\int \operatorname{cosec} x\, dx = \log|\operatorname{cosec} x - \cot x| = F(x)$

By second fundamental theorem of calculus, we obtain

$$I = F\left(\frac{\pi}{4}\right) - F\left(\frac{\pi}{6}\right)$$
$$= \log\left|\operatorname{cosec}\frac{\pi}{4} - \cot\frac{\pi}{4}\right| - \log\left|\operatorname{cosec}\frac{\pi}{6} - \cot\frac{\pi}{6}\right|$$
$$= \log\left|\sqrt{2} - 1\right| - \log\left|2 - \sqrt{3}\right|$$
$$= \log\left(\frac{\sqrt{2} - 1}{2 - \sqrt{3}}\right)$$

Question 9:

$$\int_0^1 \frac{dx}{\sqrt{1 - x^2}}$$

Solution 9:

Let $I = \int_0^1 \frac{dx}{\sqrt{1 - x^2}}$

$\int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x = F(x)$

By second fundamental theorem of calculus, we obtain

I = F (1) – F (0)

$$= \sin^{-1}(1) - \sin^{-1}(0)$$
$$= \frac{\pi}{2} - 0$$
$$= \frac{\pi}{2}$$

Question 10:

$$\int_0^1 \frac{dx}{1+x^2}$$

Solution 10:

Let $I = \int_0^1 \frac{dx}{1+x^2}$

$$\int \frac{dx}{1+x^2} = \tan^{-1} = F(x)$$

By second fundamental theorem of calculus, we obtain

I = F (1) – F (0)

$$= \tan^{-1}(1) - \tan^{-1}(0)$$

$$= \frac{\pi}{4}$$

Question 11:

$$\int_2^3 \frac{dx}{x^2-1}$$

Solution 11:

Let $I = \int_2^3 \frac{dx}{x^2-1}$

$$\int \frac{dx}{x^2-1} = \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| = F(x)$$

By second fundamental theorem of calculus, we obtain

I =F (3) -F (2)

$$= \frac{1}{2}\left[\log\left|\frac{3-1}{3+1}\right| - \log\left|\frac{2-1}{2+1}\right|\right]$$

$$= \frac{1}{2}\left[\log\left|\frac{2}{4}\right| - \log\left|\frac{1}{3}\right|\right]$$

$$= \frac{1}{2}\left[\log\frac{1}{2} - \log\frac{1}{3}\right]$$

$$= \frac{1}{2}\left[\log\frac{3}{2}\right]$$

Question 12:

$$\int_0^{\frac{\pi}{2}} \cos^2 x dx$$

Solution 12:

Let $I=\int_0^{\frac{\pi}{2}}\cos^2 x dx$

$$\int \cos^2 x dx=\int\left(\frac{1+\cos 2x}{2}\right)dx=\frac{x}{2}+\frac{\sin 2x}{4}=\frac{1}{2}\left(x+\frac{\sin 2x}{2}\right)=F(x)$$

By second fundamental theorem of calculus, we obtain

$$I=\left[F\left(\frac{\pi}{2}\right)-F(0)\right]$$

$$=\frac{1}{2}\left[\left(\frac{\pi}{2}+\frac{\sin\pi}{2}\right)-\left(0+\frac{\sin 0}{2}\right)\right]$$

$$=\frac{1}{2}\left[\frac{\pi}{2}+0-0-0\right]$$

$$=\frac{\pi}{4}$$

Question 13:

$$\int_2^3 \frac{xdx}{x^2+1}$$

Solution 13:

Let $I=\int_2^3 \frac{x}{x^2+1}dx$

$$\int \frac{x}{x^2+1}dx=\frac{1}{2}\int\frac{2x}{x^2+1}dx=\frac{1}{2}\log\left(1+x^2\right)=F(x)$$

By second fundamental theorem of calculus, we obtain

I = F (3) – F (2)

$$=\frac{1}{2}\left[\log\left(1+(3)^2\right)-\log\left(1+(2)^2\right)\right]$$

$$=\frac{1}{2}\left[\log(10)-\log(5)\right]$$

$$=\frac{1}{2}\log\left(\frac{10}{5}\right)=\frac{1}{2}\log 2$$

Question 14:

$$\int_0^1 \frac{2x+3}{5x^2+1}dx$$

Solution 14:

Let $I=\int_0^1 \frac{2x+3}{5x^2+1}dx$

$$\int \frac{2x+3}{5x^2+1}dx = \frac{1}{5}\int \frac{5(2x+3)}{5x^2+1}dx$$

$$= \frac{1}{5}\int \frac{10x+15}{5x^2+1}dx$$

$$= \frac{1}{5}\int \frac{10x}{5x^2+1}dx + 3\int \frac{1}{5x^2+1}dx$$

$$= \frac{1}{5}\int \frac{10x}{5x^2+1}dx + 3\int \frac{1}{5\left(x^2+\frac{1}{5}\right)}dx$$

$$= \frac{1}{5}\log\left(5x^2+1\right) + \frac{3}{5}\cdot\frac{1}{\frac{1}{\sqrt{5}}}\tan^{-1}\frac{x}{\frac{1}{\sqrt{5}}}$$

$$= \frac{1}{5}\log\left(5x^2+1\right) + \frac{3}{\sqrt{5}}\tan^{-1}\left(\sqrt{5}\right)x$$

$$= F(x)$$

By second fundamental theorem of calculus, we obtain

I = F (1) – F (0)

$$= \left\{\frac{1}{5}\log(5+1) + \frac{3}{\sqrt{5}}\tan^{-1}(\sqrt{5})\right\} - \left\{\frac{1}{5}\log(1) + \frac{3}{\sqrt{5}}\tan^{-1}(0)\right\}$$

$$= \frac{1}{5}\log 6 + \frac{3}{\sqrt{5}}\tan^{-1}\sqrt{5}$$

Question 15:

$$\int_0^1 xe^{x^2}dx$$

Solution 15:

Let $I = \int_0^1 xe^{x^2}dx$

Put $x^2 = t \Rightarrow 2xdx = dt$

As $x \to 0, t \to 0$ *and as* $x \to 1, t \to 1$,

$$\therefore I = \frac{1}{2}\int_0^1 e^t dt$$

$$\frac{1}{2}\int e^t dt = \frac{1}{2}e^t = F(t)$$

By second fundamental theorem of calculus, we obtain

I = F (1) – F (0)

$$= \frac{1}{2}e - \frac{1}{2}e^0$$

$$= \frac{1}{2}(e-1)$$

Question 16:

$$\int_0^1 \frac{5x^2}{x^2+4x+3}$$

Solution 16:

Let $I = \int_1^2 \frac{5x^2}{x^2+4x+3}$ dx

Dividing $5x^2$ by x^2+4x+3, we obtain

$$I = \int_1^2 \left\{5 - \frac{20x+15}{x^2+4x+3}\right\} dx$$

$$= \int_1^2 5dx - \int_1^2 \frac{20x+15}{x^2+4x+3} dx$$

$$= [5x]_1^2 - \int_1^2 \frac{20x+15}{x^2+4x+3} dx$$

$I = 5 - I_1$, where $I_1 = \int_1^2 \frac{20x+15}{x^2+4x+3} dx$ … (1)

Consider

Let $20x+15 = A\frac{d}{dx}\left(x^2+4x+3\right)+B$

$= 2Ax + (4A+B)$

Equating the coefficients of x and constant term, we obtain

A = 10 and B = −25

Let x^2+4x+3 = t

$\Rightarrow (2x+4)dx = dt$

$$\Rightarrow I_1 = 10\int \frac{dt}{t} - 25\int \frac{dx}{(x+2)^2 - 1^2}$$

$$= 10\log t - 25\left[\frac{1}{2}\log\left(\frac{x+2-1}{x+2-1}\right)\right]$$

$$= \left[10\log\left(x^2+4x+3\right)\right]_1^2 - 25\left[\frac{1}{2}\log\left(\frac{x+1}{x+3}\right)\right]_1^2$$

$$= [10\log 15 - 10\log 8] - 25\left[\frac{1}{2}\log\frac{3}{5} - \frac{1}{2}\log\frac{2}{4}\right]$$

$$= [10\log(5\times 3) - 10\log(4\times 2)] - \frac{25}{2}[\log 3 - \log 5 - \log 2 + \log 4]$$

$$= [10\log 5 + 10\log 3 - 10\log 4 - 10\log 2] - \frac{25}{2}[\log 3 - \log 5 - \log 2 + \log 4]$$

$$= \left[10 + \frac{25}{2}\right]\log 5 + \left[-10 - \frac{25}{2}\right]\log 4 + \left[10 - \frac{25}{2}\right]\log 3 + \left[-10 + \frac{25}{2}\right]\log 2$$

$$=\frac{45}{2}\log 5=\frac{45}{2}\log 4-\frac{5}{2}\log 3+\frac{5}{2}\log 2$$

$$=\frac{45}{2}\log\frac{5}{4}-\frac{5}{2}\log\frac{3}{2}$$

Substituting the value of I_1 in (1), we obtain

$$I=5-\left[\frac{45}{2}\log\frac{5}{4}-\frac{5}{2}\log\frac{3}{2}\right]$$

$$=5-\frac{5}{2}\left[9\log\frac{5}{4}-\log\frac{3}{2}\right]$$

Question 17:

$$\int_0^{\frac{\pi}{4}}\left(2\sec^2 x+x^3+2\right)dx$$

Solution 17:

Let $I=\int_0^{\frac{\pi}{4}}\left(2\sec^2 x+x^3+2\right)dx$

$$\int\left(2\sec^2 x+x^3+2\right)dx=2\tan x+\frac{x^4}{4}+2x=F(x)$$

By second fundamental theorem of calculus, we obtain

$$I=F\left(\frac{\pi}{4}\right)-F(0)$$

$$=\left\{\left(2\tan\frac{\pi}{4}+\frac{1}{4}\left(\frac{\pi}{4}\right)^4+2\left(\frac{\pi}{4}\right)\right)-(2\tan 0+0+0)\right\}$$

$$=2\tan\frac{\pi}{4}+\frac{\pi^4}{4^5}+\frac{\pi}{2}$$

$$=2+\frac{\pi}{2}+\frac{\pi^4}{1024}$$

Question 18:

$$\int_0^{\pi}\left(\sin^2\frac{x}{2}-\cos^2\frac{x}{2}\right)dx$$

Solution 18:

Let $I=\int_0^{\pi}\left(\sin^2\frac{x}{2}-\cos^2\frac{x}{2}\right)dx$

$$=-\int_0^{\pi}\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)dx$$

$$=-\int_0^{\pi}\cos x\,dx$$

$$-\int_0^{\pi} \cos x \, dx = -\sin x = F(x)$$

By second fundamental theorem of calculus, we obtain

$$I = F(\pi) - F(0)$$

$$= -\sin \pi + \sin 0$$

= 0

Question 19:

$$\int_0^2 \frac{6x+3}{x^2+4} dx$$

Solution 19:

Let $I = \int_0^2 \frac{6x+3}{x^2+4} dx$

$$\int \frac{6x+3}{x^2+4} dx = 3\int \frac{2x+1}{x^2+4} dx$$

$$= 3\int \frac{2x}{x^2+4} dx + 3\int \frac{1}{x^2+4} dx$$

$$= 3\log(x^2+4) + \frac{3}{2}\tan^{-1}\frac{x}{2} = F(x)$$

By second fundamental theorem of calculus, we obtain

I = F(2) – F(0)

$$= \left\{3\log(2^2+4) + \frac{3}{2}\tan^{-1}\left(\frac{2}{2}\right)\right\} - \left\{3\log(0+4) + \frac{3}{2}\tan^{-1}\left(\frac{0}{2}\right)\right\}$$

$$= 3\log 8 + \frac{3}{2}\tan^{-1}1 - 3\log 4 - \frac{3}{2}\tan^{-1}0$$

$$= 3\log 8 + \frac{3}{2}\left(\frac{\pi}{4}\right) - 3\log 4 - 0$$

$$= 3\log\left(\frac{8}{4}\right) + \frac{3\pi}{8}$$

$$= 3\log 2 + \frac{3\pi}{8}$$

Question 20:

$$\int_0^1 \left(xe^x + \sin\frac{\pi x}{4}\right) dx$$

Solution 20:

Let $I = \int_0^1 \left(xe^x + \sin\frac{\pi x}{4}\right) dx$

$$\int_0^1 \left(xe^x + \sin\frac{\pi x}{4}\right)dx = x\int e^x dx - \int\left\{\left(\frac{d}{dx}x\right)\int e^x dx\right\}dx + \left\{\frac{-\cos\frac{\pi x}{4}}{\frac{\pi}{4}}\right\}$$

$$= xe^x - \int e^x dx - \frac{4}{\pi}\cos\pi\frac{x}{4}$$

$$= xe^x - e^x - \frac{4}{\pi}\cos\pi\frac{x}{4}$$

$$= F(x)$$

By second fundamental theorem of calculus, we obtain

I = F (1) – F (0)

$$= \left(1.e^1 - e^1 - \frac{4}{\pi}\cos\frac{\pi}{4}\right) - \left(0.e^0 - e^0 - \frac{4}{\pi}\cos 0\right)$$

$$= e - e - \frac{4}{\pi}\left(\frac{1}{\sqrt{2}}\right) + 1 + \frac{4}{\pi}$$

$$= 1 + \frac{4}{\pi} - \frac{2\sqrt{2}}{\pi}$$

Question 21:

$$\int_1^{\sqrt{3}} \frac{dx}{1+x^2}$$

A. $\frac{\pi}{3}$

B. $\frac{2\pi}{3}$

C. $\frac{\pi}{6}$

D. $\frac{\pi}{12}$ equals

Solution 21:

$$\int \frac{dx}{1+x^2} = \tan^{-1}x = F(x)$$

By second fundamental theorem of calculus, we obtain

$$\int_1^{\sqrt{3}} \frac{dx}{1+x^2} = F\left(\sqrt{3}\right) - F(1)$$

$$= \tan^{-1}\sqrt{3} - \tan^{-1}1$$

$$= \frac{\pi}{3} - \frac{\pi}{4}$$

$$= \frac{\pi}{12}$$

Hence, the correct Answer is D.

Question 22:

$\int_0^{\frac{2}{3}} \frac{dx}{4+9x^2}$

A. $\frac{\pi}{6}$

B. $\frac{\pi}{12}$

C. $\frac{\pi}{24}$

D. $\frac{\pi}{4}$ equals

Solution 22:

$\int \frac{dx}{4+9x^2} = \int \frac{dx}{(2)^2+(3x)^2}$

Put 3x=t $\Rightarrow$ 3dx=dt

$\therefore \int \frac{dx}{(2)^2+(3x)^2} = \frac{1}{3}\int \frac{dt}{(2)^2+t^2}$

$= \frac{1}{3}\left[\frac{1}{2}\tan^{-1}\frac{t}{2}\right]$

$= \frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right)$

$= F(x)$

By second fundamental theorem of calculus, we obtain

$\int_0^{\frac{2}{3}} \frac{dx}{4+9x^2} = F\left(\frac{2}{3}\right) - F(0)$

$= \frac{1}{6}\tan^{-1}\left(\frac{3}{2}.\frac{2}{3}\right) - \frac{1}{6}\tan^{-1}0$

$= \frac{1}{6}\tan^{-1}1 - 0$

$= \frac{1}{6} \times \frac{\pi}{4}$

$= \frac{\pi}{24}$

Hence, the correct Answer is C.

Exercise 7.10

Question 1:

$\int_0^1 \frac{x}{x^2+1}dx$

Solution 1:

$\int_0^1 \frac{x}{x^2+1}dx$

Let $x^2+1=t \Rightarrow 2xdx=dt$

When x = 0, t = 1 and when x = 1, t = 2

$\therefore \int_0^1 \frac{x}{x^2+1}dx = \frac{1}{2}\int_1^2 \frac{dt}{t}$

$= \frac{1}{2}\left[\log|t|\right]_1^2$

$= \frac{1}{2}[\log 2 - \log 1]$

$= \frac{1}{2}\log 2$

Question 2:

$\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi d\phi$

Solution 2:

Let

$I = \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi d\phi$

Also, let $\sin\phi = t \Rightarrow \cos\phi d\phi = dt$

When $\phi = 0, t = 0$ and when $\phi = \frac{\pi}{2}, t = 1$

$\therefore I = \int_0^1 \sqrt{t}(1-t^2)^2 dt$

$= \int_0^1 t^{\frac{1}{2}}\left(1+t^4-2t^2\right)dt$

$= \int_0^1 \left[t^{\frac{1}{2}} + t^{\frac{9}{2}} - 2t^{\frac{5}{2}}\right]dt$

$= \left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + \frac{t^{\frac{11}{2}}}{\frac{11}{2}} - \frac{2t^{\frac{7}{2}}}{\frac{7}{2}}\right]_0^1$

$= \frac{2}{3} + \frac{2}{11} - \frac{4}{7}$

$= \frac{154+42-132}{231}$

$$=\frac{64}{231}$$

Question 3:

$$\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$$

Solution 3:

Let $I=\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

Also, let x = tanθ ⇒ dx = $\sec^2$ θ dθ

When x = 0, θ = 0 and when x = 1, $\theta=\frac{\pi}{4}$

$$I=\int_0^{\frac{\pi}{4}} \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\sec^2\theta d\theta$$

$$=\int_0^{\frac{\pi}{4}} \sin^{-1}(\sin 2\theta)\sec^2\theta d\theta$$

$$=\int_0^{\frac{\pi}{4}} 2\theta.\sec^2\theta d\theta$$

$$=2\int_0^{\frac{\pi}{4}} \theta.\sec^2\theta d\theta$$

Taking θ as first function and $\sec^2$ θ as second function and integrating by parts, we obtain

$$I=2\left[\theta\int\sec^2\theta d\theta-\int\left\{\left(\frac{d}{dx}\theta\right)\int\sec^2\theta d\theta\right\}d\theta\right]_0^{\frac{\pi}{4}}$$

$$=2\left[\theta\tan\theta-\int\tan\theta d\theta\right]_0^{\frac{\pi}{4}}$$

$$=2\left[\theta\tan\theta+\log|\cos\theta|\right]_0^{\frac{\pi}{4}}$$

$$=2\left[\frac{\pi}{4}\tan\frac{\pi}{4}+\log\left|\cos\frac{\pi}{4}\right|-\log|\cos 0|\right]$$

$$=2\left[\frac{\pi}{4}+\log\left(\frac{1}{\sqrt{2}}\right)-\log 1\right]$$

$$=2\left[\frac{\pi}{4}-\frac{1}{2}\log 2\right]$$

$$=\frac{\pi}{2}-\log 2$$

Question 4:

$\int_0^2 x\sqrt{x+2}\left(\text{Put x+2=t}^2\right)$

Solution 4:

$\int_0^2 x\sqrt{x+2}dx$

Let x + 2 = t^2 ⇒ dx = 2t dt

When x = 0, $t=\sqrt{2}$ and when x = 2, t = 2

$\therefore \int_0^2 x\sqrt{x+2}dx = \int_{\sqrt{2}}^2 \left(t^2-2\right)\sqrt{t^2}\,2dt$

$= \int_{\sqrt{2}}^2 \left(t^2-2\right)t\,dt$

$= 2\int_{\sqrt{2}}^2 \left(t^3-2t\right)dt = 2\left[\frac{t^4}{4}-t^2\right]_{\sqrt{2}}^2 = 2[(4-4)-(1-2)] = 2$

Question 5:

$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}dx$

Solution 5:

$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}dx$

Let cos x = t ⇒ −sinx dx = dt

When x = 0, t = 1 and when $x=\frac{\pi}{2}, t=0$

$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}dx = -\int_1^0 \frac{dt}{1+t^2}$

$= -\left[\tan^{-1} t\right]_1^0$

$= -\left[\tan^{-1} 0 - \tan^{-1} 1\right]$

$= -\left[-\frac{\pi}{4}\right]$

$= \frac{\pi}{4}$

Question 6:

$\int_0^2 \frac{dx}{x+4-x^2}$

Solution 6:

$$\int_0^2 \frac{dx}{x+4-x^2} = \int_0^2 \frac{dx}{-\left(x^2-x-4\right)}$$

$$= \int_0^2 \frac{dx}{-\left(x^2-x+\frac{1}{4}-\frac{1}{4}-4\right)}$$

$$= \int_0^2 \frac{dx}{-\left[\left(x-\frac{1}{2}\right)^2-\frac{17}{4}\right]}$$

$$= \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}$$

Let x-$\frac{1}{2}=t$ so dx=dt

when x=0, t=-$\frac{1}{2}$ and when x=2, t=$\frac{3}{2}$

$$\therefore \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2} = \int_{-\frac{1}{2}}^{\frac{3}{2}} \frac{dt}{\left(\frac{\sqrt{17}}{2}\right)^2-t^2}$$

$$= \left[\frac{1}{2\left(\frac{\sqrt{17}}{2}\right)} \log \frac{\frac{\sqrt{17}}{2}+t}{\frac{\sqrt{17}}{2}-t}\right]_{-\frac{1}{2}}^{\frac{3}{2}}$$

$$= \frac{1}{\sqrt{17}}\left[\log \frac{\frac{\sqrt{17}}{2}+\frac{3}{2}}{\frac{\sqrt{17}}{2}-\frac{3}{2}} - \frac{\log \frac{\sqrt{17}}{2}-\frac{1}{2}}{\log \frac{\sqrt{17}}{2}+\frac{1}{2}}\right]$$

$$= \frac{1}{\sqrt{17}}\left[\log \frac{\sqrt{17}+3}{\sqrt{17}-3} - \log \frac{\sqrt{17}-1}{\sqrt{17}+1}\right]$$

$$= \frac{1}{\sqrt{17}} \log \frac{\sqrt{17}+3}{\sqrt{17}-3} \times \frac{\sqrt{17}+1}{\sqrt{17}-1}$$

$$= \frac{1}{\sqrt{17}} \log \left[\frac{17+3+4\sqrt{17}}{17+3-4\sqrt{17}}\right]$$

$$= \frac{1}{\sqrt{17}} \log \left[\frac{20+4\sqrt{17}}{20-4\sqrt{17}}\right]$$

$$=\frac{1}{\sqrt{17}}\log\left(\frac{5+\sqrt{17}}{5-\sqrt{17}}\right)$$

$$=\frac{1}{\sqrt{17}}\log\left[\frac{\left(5+\sqrt{17}\right)\left(5+\sqrt{17}\right)}{25-17}\right]$$

$$=\frac{1}{\sqrt{17}}\log\left[\frac{25+17+10\sqrt{17}}{8}\right]$$

$$=\frac{1}{\sqrt{17}}\log\left(\frac{42+10\sqrt{17}}{8}\right)$$

$$=\frac{1}{\sqrt{17}}\log\left(\frac{21+5\sqrt{17}}{4}\right)$$

Question 7:

$$\int_{-1}^{1}\frac{dx}{x^2+2x+5}$$

Solution 7:

$$\int_{-1}^{1}\frac{dx}{x^2+2x+5}=\int_{-1}^{1}\frac{dx}{(x^2+2x+1)+4}=\int_{-1}^{1}\frac{dx}{(x+1)^2+(2)^2}$$

Let x + 1 = t ⇒ dx = dt

When x = −1, t = 0 and when x = 1, t = 2

$$\int_{-1}^{1}\frac{dx}{(x+1)^2+(2)^2}=\int_{0}^{2}\frac{dx}{t^2+2^2}$$

$$=\left[\frac{1}{2}\tan^{-1}\frac{t}{2}\right]_0^2$$

$$=\frac{1}{2}\tan^{-1}1-\frac{1}{2}\tan^{-1}0$$

$$=\frac{1}{2}\left(\frac{\pi}{4}\right)=\frac{\pi}{8}$$

Question 8:

$$\int_{1}^{2}\left(\frac{1}{x}-\frac{1}{2x^2}\right)e^{2x}dx$$

Solution 8:

$$\int_{1}^{2}\left(\frac{1}{x}-\frac{1}{2x^2}\right)e^{2x}dx$$

Let 2x = t ⇒ 2dx = dt

When x = 1, t = 2 and when x = 2, t = 4

$$\int_1^2\left(\frac{1}{x}-\frac{1}{2x^2}\right)e^{2x}dx=\frac{1}{2}\int_2^4\left(\frac{2}{t}-\frac{2}{t^2}\right)e^t dt$$

Let $\frac{1}{t}=f(t)$

Then, $f'(t)=-\frac{1}{t^2}$

$$=\int_2^4\left(\frac{1}{t}-\frac{1}{t^2}\right)e^t dt=\int_2^4 (f(t)+f'(t))e^t dt$$

$$=[e^t f(t)]_2^4$$

$$=\left[e^t\cdot\frac{1}{t}\right]_2^4$$

$$=\left[\frac{e^t}{t}\right]_2^4$$

$$=\frac{e^4}{4}-\frac{e^2}{2}$$

$$=\frac{e^2\left(e^2-2\right)}{4}$$

Question 9:

The value of the integral $\int_{\frac{1}{3}}^{1}\frac{\left(x-x^3\right)^{\frac{1}{3}}}{x^4}dx$ is

A. 6
B. 0
C. 3
D. 4

Solution 9:

Let $I=\int_{\frac{1}{3}}^{1}\frac{\left(x-x^3\right)^{\frac{1}{3}}}{x^4}dx$

Also, let x = sin $\theta \Rightarrow$ dx = cos θ dθ

When $x=\frac{1}{3},\theta=\sin^{-1}\left(\frac{1}{3}\right)$ and when x=1, $\theta=\frac{\pi}{2}$

$$\Rightarrow I=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\frac{(\sin\theta-\sin^3\theta)^{\frac{1}{3}}}{\sin^4\theta}\cos\theta d\theta$$

$$=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\frac{(\sin\theta)^{\frac{1}{3}}\left(1-\sin^2\theta\right)^{\frac{1}{3}}}{\sin^4\theta}\cos\theta d\theta$$

$$=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\frac{(\sin\theta)^{\frac{1}{3}}(\cos\theta)^{\frac{2}{3}}}{\sin^4\theta}\cos\theta d\theta$$

$$=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\frac{(\sin\theta)^{\frac{1}{3}}(\cos\theta)^{\frac{2}{3}}}{\sin^2\theta\sin^2\theta}\cos\theta d\theta$$

$$=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\frac{(\cos\theta)^{\frac{5}{3}}}{(\sin\theta)^{\frac{5}{3}}}\cos ec^2\theta d\theta$$

$$=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}(\cot\theta)^{\frac{5}{3}}\cos ec^2\theta d\theta$$

Let $\cot\theta = t \Rightarrow -\text{cosec}^2\,\theta\, d\theta = dt$

When $\theta=\sin^{-1}\left(\frac{1}{3}\right), t=2\sqrt{2}$ and when $\theta=\frac{\pi}{2}, t=0$

$$\therefore I=-\int_{2\sqrt{2}}^{0}(t)^{\frac{5}{3}}dt$$

$$=-\left[\frac{3}{8}(t)^{\frac{8}{3}}\right]_{2\sqrt{2}}^{0}$$

$$=-\frac{3}{8}\left[-\left(2\sqrt{2}\right)^{\frac{8}{3}}\right]_{2\sqrt{2}}^{0}$$

$$=\frac{3}{8}\left[\left(\sqrt{8}\right)^{\frac{8}{3}}\right]$$

$$=\frac{3}{8}\left[(8)^{\frac{4}{3}}\right]$$

$$=\frac{3}{8}[16]$$

$$=3\times 2$$

$$=6$$

Hence, the correct Answer is A.

Question 10:

If $f(x)=\int_0^x t\sin t dt,$ then $f'(x)$ is

A. cos x + x sin x

B. x sin x

C. x cos x

D. sin x + x cos x

Solution 10:

$$f(x)=\int_0^x t\sin t dt$$

Integrating by parts, we obtain

$$f(x)=t\int_0^x \sin t dt-\int_0^x\left\{\left(\frac{d}{dt}t\right)\int \sin t dt\right\}dt$$

$$=\left[t(-\cos t)\right]_0^x-\int_0^x(-\cot t)dt$$

$$=\left[-t\cos t+\sin t\right]_0^x$$

$$=-x\cos x+\sin x$$

$$\Rightarrow f'(x)=-\left[\{x(-\sin x)\}+\cos x\right]+\cos x$$

$$=x\sin x-\cos x+\cos x$$

$$=x\sin x$$

Hence, the correct Answer is B.

Exercise 7.11

Question 1:

$$\int_0^{\frac{\pi}{2}}\cos^2 x dx$$

Solution 1:

$$I=\int_0^{\frac{\pi}{2}}\cos^2 x dx \qquad ...(1)$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\cos^2\left(\frac{\pi}{2}-x\right)dx \qquad \left(\int_0^0 f(x)dx=\int_0^0 f(a-x)dx\right)$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\sin^2 x dx \qquad ...(2)$$

Adding (1) and (2), we obtain

$$2I=\int_0^{\frac{\pi}{2}}\left(\sin^2 x+\cos^2 x\right)dx$$

$$\Rightarrow 2I=\int_0^{\frac{\pi}{2}}1.dx$$

$$\Rightarrow 2I=[x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I=\frac{\pi}{2}$$

$$\Rightarrow I=\frac{\pi}{4}$$

Question 2:

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$$

Solution 2:

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$$

$$\text{Let } I=\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx \qquad ...(1)$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}dx \qquad \left(\int_0^a f(x)dx=\int_0^a f(a-x)dx\right)$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx \qquad .... (2)$$

Adding (1) and (2), we obtain

$$2I=\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$$

$$\Rightarrow 2I=\int_0^{\frac{\pi}{2}} 1.dx$$

$$\Rightarrow 2I=[x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I=\frac{\pi}{2}$$

$$\Rightarrow I=\frac{\pi}{4}$$

Question 3:

$$\int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} xdx}{\sin^{\frac{3}{2}} x+\cos^{\frac{3}{2}} x}dx$$

Solution 3:

$$\text{Let } I=\int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} xdx}{\sin^{\frac{3}{2}} x+\cos^{\frac{3}{2}} x}dx \qquad ...(1)$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}}\left(\frac{\pi}{2}-x\right)}{\sin^{\frac{3}{2}}\left(\frac{\pi}{2}-x\right)+\cos^{\frac{3}{2}}\left(\frac{\pi}{2}-x\right)}dx \qquad \left(\int_0^a f(x)dx=\int_0^a f(a-x)dx\right)$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}} \frac{\cos^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x+\cos^{\frac{3}{2}} x}dx \qquad ...(2)$$

Adding (1) and (2), we obtain

$$2I=\int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}}+\cos^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x+\cos^{\frac{3}{2}} x}dx$$

$$\Rightarrow 2I=\int_0^{\frac{\pi}{2}} 1.dx$$

$$\Rightarrow 2I=[x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I=\frac{\pi}{2}$$

$$\Rightarrow I=\frac{\pi}{4}$$

Question 4:

$$\int_0^{\frac{\pi}{2}} \frac{\cos^5 xdx}{\sin^5 x+\cos^5 x}dx$$

Solution 4:

Let $I=\int_0^{\frac{\pi}{2}} \frac{\cos^5 xdx}{\sin^5 x+\cos^5 x}dx \qquad ...(1)$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}} \frac{\cos^5\left(\frac{\pi}{2}-x\right)}{\sin^5\left(\frac{\pi}{2}-x\right)+\cos^5\left(\frac{\pi}{2}-x\right)}dx \qquad \left(\int_0^a f(x)dx=\int_0^a f(a-x)dx\right)$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}} \frac{\sin^5 x}{\sin^5 x+\cos^5 x}dx \qquad ...(2)$$

Adding (1) and (2), we obtain

$$2I=\int_0^{\frac{\pi}{2}} \frac{\sin^5 x+\cos^5 x}{\sin^5 x+\cos^5 x}dx$$

$$\Rightarrow 2I=\int_0^{\frac{\pi}{2}} 1.dx$$

$$\Rightarrow 2I=[x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

Question 5:

$$\int_{-5}^{5} |x+2| dx$$

Solution 5:

Let $I = \int_{-5}^{5} |x+2| dx$

It can be seen that $(x + 2) \le 0$ on $[-5, -2]$ and $(x + 2) \ge 0$ on $[-2, 5]$.

$$\therefore I = \int_{-5}^{-2} -(x+2)\, dx + \int_{-2}^{5} (x+2)\, dx$$

$$I = -\left[\frac{x^2}{2} + 2x\right]_{-5}^{-2} + \left[\frac{x^2}{2} + 2x\right]_{-2}^{5}$$

$$= -\left[\frac{(-2)^2}{2} + 2(-2) - \frac{(-5)^2}{2} - 2(-5)\right] + \left[\frac{(5)^2}{2} + 2(5) - \frac{(-2)^2}{2} - 2(-2)\right]$$

$$= -\left[2 - 4 - \frac{25}{2} + 10\right] + \left[\frac{25}{2} + 10 - 2 + 4\right]$$

$$= -2 + 4 + \frac{25}{2} - 10 + \frac{25}{2} + 10 - 2 + 4$$

$$= 29$$

Question 6:

$$\int_{2}^{8} |x-5| dx$$

Solution 6:

Let $I = \int_{2}^{8} |x-5| dx$

It can be seen that $(x - 5) \le 0$ on $[2, 5]$ and $(x - 5) \ge 0$ on $[5, 8]$.

$$I = \int_{2}^{5} -(x-5)\, dx + \int_{2}^{8} (x-5)\, dx \qquad \left(\int_a^b f(x) = \int_a^c f(x) + \int_c^b f(x)\right)$$

$$= -\left[\frac{x^2}{2} - 5x\right]_{2}^{5} + \left[\frac{x^2}{2} - 5x\right]_{5}^{8}$$

$$= -\left[\frac{25}{2} - 25 - 2 + 10\right] + \left[32 - 40 - \frac{25}{2} + 25\right]$$

$$= 9$$

Question 7:

$$\int_0^1 x(1-x)^n\,dx$$

Solution 7:

Let $I=\int_0^1 x(1-x)^n\,dx$

$$\therefore I=\int_0^1 (1-x)\left(1-(1-x)\right)^n dx \qquad \left(\int_1^a f(x)dx=\int_0^a f(a-x)dx\right)$$

$$=\int_0^1 (1-x)(x)^n dx$$

$$=\int_0^1 \left(x^n-x^{n+1}\right)dx$$

$$=\left[\frac{x^{n+1}}{n+1}-\frac{x^{n+2}}{n+2}\right]_0^1$$

$$=\left[\frac{1}{n+1}-\frac{1}{n+2}\right]$$

$$=\frac{(n+2)-(n+1)}{(n+1)(n+2)}$$

$$=\frac{1}{(n+1)(n+2)}$$

Question 8:

$$\int_0^{\frac{\pi}{4}} \log(1+\tan x)\,dx$$

Solution 8:

Let $I=\int_0^{\frac{\pi}{4}} \log(1+\tan x)\,dx$ ...(1)

$$\therefore I=\int_0^{\frac{\pi}{4}} \log\left[1+\tan\left(\frac{\pi}{4}-x\right)\right]dx \qquad \left(\int_0^a f(x)dx=\int_0^a f(a-x)dx\right)$$

$$\Rightarrow I=\int_0^{\frac{\pi}{4}} \log\left\{1+\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right\}dx$$

$$\Rightarrow I=\int_0^{\frac{\pi}{4}} \log\left\{1+\frac{1-\tan x}{1+\tan x}\right\}dx$$

$$\Rightarrow I=\int_0^{\frac{\pi}{4}} \log\frac{2}{(1+\tan x)}dx$$

$$\Rightarrow I=\int_0^{\frac{\pi}{4}} \log 2dx-\int_0^{\frac{\pi}{4}} \log(1+\tan x)\,dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log 2 dx - I \qquad \left[From(1)\right]$$

$$\Rightarrow 2I = \left[x \log 2\right]_0^{\frac{\pi}{4}}$$

$$\Rightarrow 2I = \frac{\pi}{4} \log 2$$

$$\Rightarrow I = \frac{\pi}{8} \log 2$$

Question 9:

$$\int_0^2 x\sqrt{2-x}dx$$

Solution 9:

Let $I = \int_0^2 x\sqrt{2-x}dx$

$$I = \int_0^2 (2-x)\sqrt{x}dx \qquad \left(\int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$$

$$= \int_0^2 \left\{2x^{\frac{1}{2}} - x^{\frac{3}{2}}\right\}dx$$

$$= \left[2\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right) - \frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right]_0^2$$

$$= \left[\frac{4}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}}\right]_0^2$$

$$= \frac{4}{3}(2)^{\frac{3}{2}} - \frac{2}{5}(2)^{\frac{5}{2}}$$

$$= \frac{4\times 2\sqrt{2}}{3} - \frac{2}{5}\times 4\sqrt{2}$$

$$= \frac{8\sqrt{2}}{3} - \frac{8\sqrt{2}}{5}$$

$$= \frac{40\sqrt{2} - 24\sqrt{2}}{15}$$

$$= \frac{16\sqrt{2}}{15}$$

Question 10:

$$\int_0^{\frac{\pi}{2}} (2\log \sin x - \log \sin 2x)dx$$

Solution 10:

Let $I=\int_0^{\frac{\pi}{2}}(2\log\sin x-\log\sin 2x)\,dx$

$\Rightarrow I=\int_0^{\frac{\pi}{2}}\{2\log\sin x-\log(2\sin x\cos x)\}\,dx$

$\Rightarrow I=\int_0^{\frac{\pi}{2}}\{2\log\sin x-\log\sin x-\log\cos x-\log 2\}dx$

$\Rightarrow I=\int_0^{\frac{\pi}{2}}\{\log\sin x-\log\cos x-\log 2\}\,dx \qquad ...(1)$

It is known that, $\left(\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx\right)$

$\Rightarrow I=\int_0^{\frac{\pi}{2}}\{\log\cos x-\log\sin x-\log 2\}\,dx \qquad ...(2)$

Adding (1) and (2), we obtain

$2I=\int_0^{\frac{\pi}{2}}(-\log 2-\log 2)\,dx$

$\Rightarrow 2I=-2\log 2\int_0^{\frac{\pi}{2}}1.dx$

$\Rightarrow I=-\log 2\left[\frac{\pi}{2}\right]$

$\Rightarrow I=\frac{\pi}{2}(-\log 2)$

$\Rightarrow I=\frac{\pi}{2}\left[\log\frac{1}{2}\right]$

$\Rightarrow I=\frac{\pi}{2}\log\frac{1}{2}$

Question 11:

$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\sin^2 x dx$

Solution 11:

Let $I=\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\sin^2 x dx$

As $\sin^2(-x)=(\sin(-x))^2=(-\sin x)^2=\sin^2 x$, therefore, $\sin^2 x$ is an even function.

It is known that if f(x) is an even function, then $\int_{-a}^a f(x)\,dx=2\int_0^a f(x)\,dx$

$I=2\int_0^{\frac{\pi}{2}}\sin^2 x dx$

$=2\int_0^{\frac{\pi}{2}}\frac{1-\cos 2x}{2}\,dx$

$$= \int_0^{\frac{\pi}{2}} (1 - \cos 2x)\, dx$$

$$= \left[x - \frac{\sin 2x}{2} \right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2}$$

Question 12:

$$\int_0^{\pi} \frac{x dx}{1 + \sin x}$$

Solution 12:

Let $I = \int_0^{\pi} \frac{x dx}{1 + \sin x}$ ...(1)

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)}{1 + \sin(\pi - x)} dx \qquad \left(\int_0^{\pi} f(x)\, dx = \int_0^{\pi} f(a - x)\, dx \right)$$

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)}{1 + \sin x} dx \qquad ...(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^{\pi} \frac{\pi}{1 + \sin x} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{(1 - \sin x)}{(1 + \sin x)(1 - \sin x)} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{1 - \sin x}{\cos^2 x} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \{\sec^2 x - \tan x \sec x\} dx$$

$$\Rightarrow 2I = \pi [2]$$

$$\Rightarrow I = \pi$$

Question 13:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x dx$$

Solution 13:

Let $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x dx$ ...(1)

As $\sin^7 (-x) = (\sin (-x))^7 = (-\sin x)^7 = -\sin^7 x$, therefore, $\sin^2 x$ is an odd function.

It is known that, if f(x) is an odd function, then $\int_{-a}^{a} f(x)\, dx = 0$

$$\therefore I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^7 x dx=0$$

Question 14:

$$\int_0^{2\pi}\cos^5 x dx$$

Solution 14:

Let $I=\int_0^{2\pi}\cos^5 x dx \qquad ...(1)$

$\cos^5(2\pi-x)=\cos^5 x$

It is known that,

$$\int_0^{2a} f(x)dx=2\int_0^a f(x)dx,\text{ if } f(2a-x)=f(x)$$
$$=0 \text{ if } f(2a-x)=-f(x)$$

$$\therefore I=2\int_0^{\pi}\cos^5 x dx$$

$\Rightarrow I=2(0)=0 \qquad [\cos^5(\pi-x)=-\cos^5 x]$

Question 15:

$$\int_0^{\frac{\pi}{2}}\frac{\sin x-\cos x}{1+\sin x\cos x}dx$$

Solution 15:

Let $I=\int_0^{\frac{\pi}{2}}\frac{\sin x-\cos x}{1+\sin x\cos x}dx \qquad ...(1)$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\sin\left(\frac{\pi}{2}-x\right)-\cos\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}dx \qquad \left(\int_0^a f(x)dx=\int_0^a f(a-x)dx\right)$$

$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\cos x-\sin x}{1+\sin x\cos x}dx \qquad ...(2)$

Adding (1) and (2), we obtain

$$\Rightarrow 2I=\int_0^{\frac{\pi}{2}}\frac{0}{1+\sin x\cos x}dx$$

$\Rightarrow I=0$

Question 16:

$$\int_0^{\pi}\log(1+\cos x)dx$$

Solution 16:

Let $I = \int_0^{\pi} \log(1+\cos x)\,dx \qquad ...(1)$

$\Rightarrow I = \int_0^{\pi} \log(1+\cos(\pi - x))\,dx \qquad \left(\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)$

$\Rightarrow I = \int_0^{\pi} \log(1-\cos x)\,dx \qquad ...(2)$

Adding (1) and (2), we obtain

$2I = \int_0^{\pi} \{\log(1-\cos x) + \log(1-\cos x)\}\,dx$

$\Rightarrow 2I = \int_0^{\pi} \log(1-\cos^2 x)\,dx$

$\Rightarrow 2I = \int_0^{\pi} \log \sin^2 x dx$

$\Rightarrow 2I = 2\int_0^{\pi} \log \sin x dx$

$\Rightarrow I = \int_0^{\pi} \log \sin x dx \qquad ...(3)$

sin (π − x) = sin x

$\therefore I = 2\int_0^{\frac{\pi}{2}} \log \sin x\,dx \qquad ...(4)$

$\Rightarrow I = 2\int_0^{\frac{\pi}{2}} \log \sin\left(\frac{\pi}{2} - x\right)dx = 2\int_0^{\frac{\pi}{2}} \log \cos x\,dx \qquad ...(5)$

Adding (4) and (5), we obtain

$2I = 2\int_0^{\frac{\pi}{2}} (\log \sin x + \log \cos x)\,dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} (\log \sin x + \log \cos x + \log 2 - \log 2)\,dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} (\log 2 \sin x \cos x - \log 2)\,dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \log \sin 2x dx - \int_0^{\frac{\pi}{2}} \log 2 dx$

Let 2x = t ⇒ 2dx = dt

When x = 0, t = 0 and when x = π/2, t = π

$\therefore I = \frac{1}{2}\int_0^{\pi} \log \sin t\,dt - \frac{\pi}{2}\log 2$

$\Rightarrow I = \frac{I}{2} - \frac{\pi}{2}\log 2$

$\Rightarrow \frac{I}{2} = -\frac{\pi}{2}\log 2$

$\Rightarrow I = -\pi \log 2$

Question 17:

$$\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}\,dx$$

Solution 17:

Let $I=\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}\,dx \qquad ...(1)$

It is known that, $\left(\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx\right)$

$$I=\int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}\,dx \qquad ...(2)$$

Adding (1) and (2), we obtain

$$2I=\int_0^a \frac{\sqrt{x}+\sqrt{a-x}}{\sqrt{x}+\sqrt{a-x}}\,dx$$

$$\Rightarrow 2I=\int_0^a 1.dx$$

$$\Rightarrow 2I=[x]_0^a$$

$$\Rightarrow 2I=a$$

$$\Rightarrow I=\frac{a}{2}$$

Question 18:

$$\int_0^4 |x-1|\,dx$$

Solution 18:

$$I=\int_0^4 |x-1|\,dx$$

It can be seen that, $(x-1)\le 0$ when $0\le x\le 1$ and $(x-1)\ge 0$ when $1\le x\le 4$

$$I=\int_0^1 |x-1|\,dx+\int_1^4 |x-1|\,dx \qquad \left(\int_a^b f(x)=\int_a^c f(x)+\int_c^b f(x)\right)$$

$$I=\int_0^1 -(x-1)\,dx+\int_0^4 (x-1)\,dx$$

$$=\left[x-\frac{x^2}{2}\right]_0^1+\left[\frac{x^2}{2}-x\right]_1^4$$

$$=1-\frac{1}{2}+\frac{(4)^2}{2}-4-\frac{1}{2}+1$$

$$=1-\frac{1}{2}+8-4-\frac{1}{2}+1$$

$$=5$$

Question 19:

Show that $\int_0^a f(x)g(x)dx = 2\int_0^a f(x)dx$, if *f* and *g* are defined as f(x) = f(a – x) and g (x) + g (a – x) = 4

Solution 19:

Let $\int_0^a f(x)g(x)dx$ ...(1)

$\Rightarrow \int_0^a f(a-x)g(a-x)dx$ $\left(\int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$

$\Rightarrow \int_0^a f(x)g(a-x)dx$ ...(2)

Adding (1) and (2), we obtain

$2I = \int_0^a \{f(x)g(x) + f(x)g(a-x)\}dx$

$\Rightarrow 2I = \int_0^a f(x)\{g(x)+g(a-x)\}dx$

$\Rightarrow 2I = \int_0^a f(x)\times 4dx$ $[g(x)+g(a-x)=4]$

$\Rightarrow I = 2\int_0^a f(x)dx$

Question 20:

The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(x^3 + x\cos x + \tan^5 x + 1\right)dx$ is

A. 0
B. 2
C. π
D. 1

Solution 20:

Let $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(x^3 + x\cos x + \tan^5 x + 1\right)dx$

$\Rightarrow I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos xdx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan^5 xdx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1.dx$

It is known that if f (x) is an even function, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$

if f (x) is an odd function, then $\int_{-a}^{a} f(x)dx = 0$

and $I = 0+0+0+2\int_0^{\frac{\pi}{2}} 1dx$

$= 2[x]_0^{\frac{\pi}{2}}$

$= \frac{2\pi}{2}$

$= \pi$

Hence, the correct Answer is C.

Question 21:

The value of $\int_0^{\frac{\pi}{2}} \left(\frac{4+3\sin x}{4+3\cos x} \right) dx$ is

A. 2

B. $\frac{3}{4}$

C. 0

D. –2

Solution 21:

Let I = $\int_0^{\frac{\pi}{2}} \left(\frac{4+3\sin x}{4+3\cos x} \right) dx$ …(1)

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \left[\frac{4+3\sin\left(\frac{\pi}{2}-x\right)}{4+3\cos\left(\frac{\pi}{2}-x\right)} \right] dx \qquad \left(\int_0^a f(x)dx = \int_0^a f(a-x)dx \right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\cos x}{4+3\sin x} \right) dx \qquad ...(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^{\frac{\pi}{2}} \left\{ \log\left(\frac{4+3\sin x}{4+3\cos x} \right) + \log\left(\frac{4+3\cos x}{4+3\sin x} \right) \right\} dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \left(\frac{4+3\sin x}{4+3\cos x} \times \frac{4+3\cos x}{4+3\sin x} \right) dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \log 1 dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 0 dx$$

$$\Rightarrow I = 0$$

Hence, the correct Answer is C.

Miscellaneous Exercise

Question 1:

Integrate $\frac{1}{x-x^3}$

Solution 1:

$$\frac{1}{x-x^3}=\frac{1}{x\left(1-x^2\right)}=\frac{1}{x(1-x)(1+x)}$$

Let $\frac{1}{x(1-x)(1+x)}=\frac{A}{x}+\frac{B}{(1-x)}+\frac{C}{1+x}$ ...(1)

$$\Rightarrow 1=A\left(1-x^2\right)+Bx(1+x)+Cx(1-x)$$

$$\Rightarrow 1=A-Ax^2+Bx+Bx^2+Cx-Cx^2$$

Equating the coefficients of x^2, x, and constant term, we obtain

−A + B − C = 0

B + C = 0

A = 1

On solving these equations, we obtain

$$A=1,\ B=\frac{1}{2},\ \text{and } C=-\frac{1}{2}$$

From equation (1), we obtain

$$\frac{1}{x(1-x)(1+x)}=\frac{1}{x}+\frac{1}{2(1-x)}-\frac{1}{2(1+x)}$$

$$\Rightarrow \int\frac{1}{x(1-x)(1+x)}dx=\int\frac{1}{x}dx+\frac{1}{2}\int\frac{1}{(1-x)}dx-\frac{1}{2}\int\frac{1}{(1+x)}dx$$

$$=\log|x|-\frac{1}{2}\log|(1-x)|-\frac{1}{2}\log|(1+x)|$$

$$=\log|x|-\log\left|(1-x)^{\frac{1}{2}}\right|-\log\left|(1+x)^{\frac{1}{2}}\right|$$

$$=\log\left|\frac{x}{(1-x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}}}\right|+C$$

$$=\log\left|\left(\frac{x^2}{1-x^2}\right)^{\frac{1}{2}}\right|+C$$

$$=\frac{1}{2}\log\left|\frac{x^2}{1-x^2}\right|+C$$

Question 2:

Integrate $\frac{1}{\sqrt{x+a}+\sqrt{(x+b)}}$

Solution 2:

$$\frac{1}{\sqrt{x+a}+\sqrt{(x+b)}}=\frac{1}{\sqrt{x+a}+\sqrt{x+b}}\times\frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{x+a}-\sqrt{x+b}}$$

$$=\frac{\sqrt{x+a}-\sqrt{x+b}}{(x+a)-(x-b)}$$

$$=\frac{\left(\sqrt{x+a}-\sqrt{x+b}\right)}{a-b}$$

$$\Rightarrow\int\frac{1}{\sqrt{x+a}+\sqrt{(x+b)}}dx=\frac{1}{a-b}\int\left(\sqrt{x+a}-\sqrt{x+b}\right)dx$$

$$=\frac{1}{(a-b)}\left[\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}}-\frac{(x+b)^{\frac{3}{2}}}{\frac{3}{2}}\right]$$

$$=\frac{2}{3(a-b)}\left[(x+a)^{\frac{3}{2}}-(x+b)^{\frac{3}{2}}\right]+C$$

Question 3:

Integrate $\frac{1}{x\sqrt{ax-x^2}}$ $\left[\text{Hint: } x=\frac{a}{t}\right]$

Solution 3:

$$\frac{1}{x\sqrt{ax-x^2}}$$

Let $x=\frac{a}{t}\Rightarrow dx=-\frac{a}{t^2}dh$

$$\Rightarrow\int\frac{1}{x\sqrt{ax-x^2}}dx=\int\frac{1}{\frac{a}{t}\sqrt{a\cdot\frac{a}{t}-\left(\frac{a}{t}\right)^2}}\left(-\frac{a}{t^2}dt\right)$$

$$=-\int\frac{1}{at}\cdot\frac{1}{\sqrt{\frac{1}{t}-\frac{1}{t^2}}}dt$$

$$=-\frac{1}{a}\int\frac{1}{\sqrt{t-1}}dt$$

$$=-\frac{1}{a}\left[2\sqrt{t-1}\right]+C$$

$$=-\frac{1}{a}\left[2\sqrt{\frac{a}{x}-1}\right]+C$$

$$= -\frac{2}{a}\left(\frac{\sqrt{a-x}}{\sqrt{x}}\right) + C$$

$$= -\frac{2}{a}\left(\sqrt{\frac{a-x}{x}}\right) + C$$

Question 4:

Integrate $\dfrac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}$

Solution 4:

$$\frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}$$

Multiplying and dividing by x^{-3}, we obtain

$$\frac{x^{-3}}{x^2x^{-3}(x^4+1)^{\frac{3}{4}}} = \frac{x^{-3}(x^4+1)^{\frac{-3}{4}}}{x^2x^{-3}}$$

$$= \frac{\left(x^4+1\right)^{\frac{-3}{4}}}{x^5.\left(x^4\right)^{-\frac{3}{4}}}$$

$$= \frac{1}{x^5}\left(\frac{x^4+1}{x^4}\right)^{-\frac{3}{4}}$$

$$= \frac{1}{x^5}\left(1+\frac{1}{x^4}\right)^{-\frac{3}{4}}$$

Let $\frac{1}{x^4} = t \Rightarrow -\frac{4}{x^5}dx = dt \Rightarrow \frac{1}{x^5}dx = -\frac{dt}{4}$

$$\therefore \int \frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}dx = \int \frac{1}{x^5}\left(1+\frac{1}{x^4}\right)^{-\frac{3}{4}}dx$$

$$= -\frac{1}{4}\int (1+t)^{-\frac{3}{4}}dt$$

$$= -\frac{1}{4}\left[\frac{(1+t)^{\frac{1}{4}}}{\frac{1}{4}}\right] + C$$

$$=-\frac{1}{4}\frac{\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}}}{\frac{1}{4}}+C$$

$$=-\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}}+C$$

Question 5:

Integrate $\dfrac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}$

$$\left[\text{Hint:}\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}=\frac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)}\text{ Put x = t}^6\right]$$

Solution 5:

Let $x=t^6 \Rightarrow dx=6t^5dt$

$$\int\frac{1}{x^{1/2}+x^{1/3}}dx=\int\frac{6t^5}{t^3+t^2}dt$$

$$=\int\frac{6t^5}{t^2(1+t)}dt$$

$$=6\int\frac{t^3}{(1+t)}dt$$

On dividing, we obtain

$$\int\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}dx=6\int\left\{\left(t^2-t+1\right)-\frac{1}{1+t}\right\}dt$$

$$=6\left[\left(\frac{t^3}{3}\right)-\left(\frac{t^2}{2}\right)+t-\log|1+t|\right]$$

$$=2x^{\frac{1}{2}}-3x^{\frac{1}{3}}+6x^{\frac{1}{6}}-6\log\left(1+x^{\frac{1}{6}}\right)+C$$

$$=2\sqrt{x}-3x^{\frac{1}{3}}+6x^{\frac{1}{6}}-6\log\left(1-x^{\frac{1}{6}}\right)+C$$

Question 6:

Integrate $\frac{5x}{(x+1)(x^2+9)}$

Solution 6:

Let $\frac{5x}{(x+1)(x^2+9)} = \frac{A}{(x+1)} + \frac{Bx+C}{(x^2+9)}$ ...(1)

$\Rightarrow 5x = A(x^2+9) + (Bx+C)(x+1)$

$\Rightarrow 5x = Ax^2 + 9A + Bx^2 + Bx + Cx + C$

Equating the coefficients of x^2, x, and constant term, we obtain

A + B = 0

B + C = 5

9A + C = 0

On solving these equations, we obtain

$A = -\frac{1}{2}, B = \frac{1}{2}, and\ C = \frac{9}{2}$

From equation (1), we obtain

$$\frac{5x}{(x+1)(x^2+9)} = \frac{-1}{2(x+1)} + \frac{\frac{x}{2}+\frac{9}{2}}{(x^2+9)}$$

$$\int \frac{5x}{(x+1)(x^2+9)} dx = \int \left\{ \frac{-1}{2(x+1)} + \frac{(x+9)}{2(x^2+9)} \right\} dx$$

$$= -\frac{1}{2}\log|x+1| + \frac{1}{2}\int \frac{x}{x^2+9} dx + \frac{9}{2}\int \frac{1}{x^2+9} dx$$

$$= -\frac{1}{2}\log|x+1| + \frac{1}{4}\int \frac{2x}{x^2+9} dx + \frac{9}{2}\int \frac{1}{x^2+9} dx$$

$$= -\frac{1}{2}\log|x+1| + \frac{1}{4}\log|x^2+9| + \frac{9}{2}.\frac{1}{3}\tan^{-1}\frac{x}{3}$$

$$= -\frac{1}{2}\log|x+1| + \frac{1}{4}\log(x^2+9) + \frac{3}{2}\tan^{-1}\frac{x}{3} + C$$

Question 7:

Integrate $\frac{\sin x}{\sin(x-a)}$

Solution 7:

$$\frac{\sin x}{\sin(x-a)}$$

Let x − a = t ⇒ dx = dt

$$\int \frac{\sin x}{\sin(x-a)}dx = \int \frac{\sin(t+a)}{\sin t}dt$$
$$= \int \frac{\sin t \cos a + \cos t \sin a}{\sin t}dt$$
$$= \int (\cos a + \cot t \sin a)\,dt$$
$$= t\cos a + \sin a \log\left|\sin t\right| + C_1$$
$$= (x-a)\cos a + \sin a \log\left|\sin(x-a)\right| + C_1$$
$$= x\cos a + \sin a \log\left|\sin(x-a)\right| - a\cos a + C_1$$
$$= \sin a \log\left|\sin(x-a)\right| + x\cos a + C$$

Question 8:

Integrate $\dfrac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}}$

Solution 8:

$$\frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}} = \frac{e^{4\log x}\left(e^{\log x} - 1\right)}{e^{2\log x}\left(e^{\log x} - 1\right)}$$
$$= e^{2\log x}$$
$$= e^{\log x^2}$$
$$= x^2$$
$$\therefore \int \frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}}dx = \int x^2 dx = \frac{x^3}{3} + C$$

Question 9:

Integrate $\dfrac{\cos x}{\sqrt{4-\sin^2 x}}$

Solution 9:

$$\frac{\cos x}{\sqrt{4-\sin^2 x}}$$

Let sin x = t ⇒ cos x dx = dt

$$\Rightarrow \int \frac{\cos x}{\sqrt{4-\sin^2 x}}dx = \int \frac{dt}{\sqrt{(2)^2 - (t)^2}}$$
$$= \sin^{-1}\left(\frac{t}{2}\right) + C$$
$$= \sin^{-1}\left(\frac{\sin x}{2}\right) + C$$

Question 10:

Integrate $\dfrac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}$

Solution 10:

$$\frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}=\frac{\left(\sin^4 x+\cos^4 x\right)\left(\sin^4 x-\cos^4 x\right)}{1-2\sin^2 x+\cos^2 x}$$

$$=\frac{\left(\sin^4 x+\cos^4 x\right)\left(\sin^2 x+\cos^2 x\right)\left(\sin^2 x-\cos^2 x\right)}{1-2\sin^2 x+\cos^2 x}$$

$$=\frac{\left(\sin^4 x+\cos^4 x\right)\left(\sin^2 x-\cos^2 x\right)}{1-2\sin^2 x+\cos^2 x}$$

$$=\frac{-\left(\sin^4 x+\cos^4 x\right)\left(\cos^2 x-\sin^2 x\right)}{\left(\sin^4 x+\cos^4 x\right)}$$

$$=-\cos 2x$$

$$\therefore\int\frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}dx=\int-\cos 2x dx=-\frac{\sin 2x}{2}+C$$

Question 11:

Integrate $\dfrac{1}{\cos(x+a)\cos(x+b)}$

Solution 11:

$$\frac{1}{\cos(x+a)\cos(x+b)}$$

Multiplying and dividing by sin (a – b), we obtain.

$$\frac{1}{\sin(a-b)}\left[\frac{\sin(a-b)}{\cos(x+a)\cos(x+b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\frac{\sin\left[(x+a)-(x+b)\right]}{\cos(x+a)\cos(x+b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\frac{\sin(x+a).\cos(x+b)-\cos(x+a)\sin(x+b)}{\cos(x+a)\cos(x+b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\frac{\sin(x+a)}{\cos(x+a)}-\frac{\sin(x+b)}{\cos(x+b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\tan(x+a)-\tan(x+b)\right]$$

$$\int \frac{1}{\cos(x+a)\cos(x+b)} dx = \frac{1}{\sin(a-b)} \int [\tan(x+a) - \tan(x+b)] dx$$

$$= \frac{1}{\sin(a-b)} [-\log|\cos(x+a)| + \log|\cos(x+b)|] + C$$

$$= \frac{1}{\sin(a-b)} \log\left|\frac{\cos(x+b)}{\cos(x+a)}\right| + C$$

Question 12:

Integrate $\frac{x^3}{\sqrt{1-x^8}}$

Solution 12:

$$\frac{x^3}{\sqrt{1-x^8}}$$

Let $x^4 = t \Rightarrow 4x^3\, dx = dt$

$$\Rightarrow \int \frac{x^3}{\sqrt{1-x^8}} dx = \frac{1}{4} \int \frac{dt}{\sqrt{1-t^2}}$$

$$= \frac{1}{4} \sin^{-1} t + C$$

$$= \frac{1}{4} \sin^{-1}(x^4) + C$$

Question 13:

Integrate $\frac{e^x}{(1+e^x)(2+e^x)}$

Solution 13:

$$\frac{e^x}{(1+e^x)(2+e^x)}$$

Let $e^x = t \Rightarrow e^x\, dx = dt$

$$\Rightarrow \int \frac{e^x}{(1+e^x)(2+e^x)} dx = \int \frac{dt}{(t+1)(t+2)}$$

$$= \int \left[\frac{1}{(t+1)} - \frac{1}{(t+2)}\right] dt$$

$$= \log|t+1| - \log|t+2| + C$$

$$= \log\left|\frac{t+1}{t+2}\right| + C$$

$$= \log\left|\frac{1+e^x}{2+e^x}\right| + C$$

Question 14:

Integrate $\frac{1}{\left(x^2+1\right)\left(x^2+4\right)}$

Solution 14:

$$\therefore \frac{1}{\left(x^2+1\right)\left(x^2+4\right)} = \frac{Ax+B}{\left(x^2+1\right)} + \frac{Cx+D}{\left(x^2+4\right)}$$

$$\Rightarrow 1 = (Ax+B)\left(x^2+4\right) + (Cx+D)\left(x^2+1\right)$$

$$\Rightarrow 1 = Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + Cx + Dx^2 + D$$

Equating the coefficients of x^3, x^2, x, and constant term, we obtain

A + C = 0

B + D = 0

4A + C = 0

4B + D = 1

On solving these equations, we obtain

$$A = 0,\ B = \frac{1}{3}, C = 0 \text{ and } D=-\frac{1}{3}$$

From equation (1), we obtain

$$\frac{1}{\left(x^2+1\right)\left(x^2+4\right)} = \frac{1}{3\left(x^2+1\right)} - \frac{1}{3\left(x^2+4\right)}$$

$$\int \frac{1}{\left(x^2+1\right)\left(x^2+4\right)} dx = \frac{1}{3}\int \frac{1}{x^2+1} dx - \frac{1}{3}\int \frac{1}{x^2+4} dx$$

$$= \frac{1}{3}\tan^{-1} x - \frac{1}{3}.\frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

$$= \frac{1}{3}\tan^{-1} x - \frac{1}{6}\tan^{-1}\frac{x}{2} + C$$

Question 15:

Integrate $\cos^3 x e^{\log \sin x}$

Solution 15:

$\cos^3 x e^{\log \sin x}$ = $\cos^3$ x × sin x

Let cos x = t ⇒ −sin x dx = dt

$$\Rightarrow \int \cos^3 x e^{\log \sin x} dx = \int \cos^3 x \sin x dx$$

$$= -\int t^3\, dx$$

$$= -\frac{t^4}{4} + C$$

$$= -\frac{\cos^4 x}{4} + C$$

Question 16:

Integrate $e^{3\log x}\left(x^4+1\right)^{-1}$

Solution 16:

$$e^{3\log x}\left(x^4+1\right)^{-1} = e^{\log x^3}\left(x^4+1\right)^{-1} = \frac{x^3}{\left(x^4+1\right)}$$

$$Let\ x^4+1 = t \Rightarrow 4x^3dx = dt$$

$$\Rightarrow \int e^{3\log x} = \left(x^4+1\right)^{-1} dx = \int \frac{x^3}{\left(x^4+1\right)} dx$$

$$= \frac{1}{4}\int \frac{dt}{t}$$

$$= \frac{1}{4}\log|t| + C$$

$$= \frac{1}{4}\log\left|x^4+1\right| + C$$

$$= \frac{1}{4}\log\left(x^4+1\right) + C$$

Question 17:

Integrate $f'(ax+b)\left[f(ax+b)\right]^n$

Solution 17:

$$f'(ax+b)\left[f(ax+b)\right]^n$$

Let $f(ax+b) = t \Rightarrow a\, f'(ax+b)dx = dt$

$$\Rightarrow \int f'(ax+b)\left[f(ax+b)\right]^n dx = \frac{1}{a}\int t^n dt$$

$$= \frac{1}{a}\left[\frac{t^{n+1}}{n+1}\right]$$

$$= \frac{1}{a(n+1)}\left(f(ax+b)\right)^{n+1} + C$$

Question 18:

Integrate $\dfrac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}}$

Solution 18:

$$\frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}} = \frac{1}{\sqrt{\sin^3 x(\sin x \cos\alpha + \cos x \sin\alpha)}}$$

$$= \frac{1}{\sqrt{\sin^4 x \cos\alpha + \sin^3 x \cos x \sin\alpha}}$$

$$= \frac{1}{\sin^2 x\sqrt{\cos\alpha + \cot x \sin\alpha}}$$

$$= \frac{\operatorname{cosec}^2 x}{\sqrt{\cos\alpha + \cot x \sin\alpha}}$$

Let $\cos\alpha + \cot x \sin\alpha = t \Rightarrow -\operatorname{cosec}^2 x \sin\alpha dx = dt$

$$\therefore \int \frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}} dx = \int \frac{\operatorname{cosec}^2 x}{\sqrt{\cos\alpha + \cot x \sin\alpha}} dx.$$

$$= \frac{-1}{\sin\alpha}\int \frac{dt}{\sqrt{t}}$$

$$= \frac{-1}{\sin\alpha}\left[2\sqrt{t}\right] + C$$

$$= \frac{-1}{\sin\alpha}\left[2\sqrt{\cos\alpha + \cot x \sin\alpha}\right] + C$$

$$= \frac{-2}{\sin\alpha}\sqrt{\cos\alpha + \frac{\cos x \sin\alpha}{\sin x}} + C$$

$$= \frac{-2}{\sin\alpha}\sqrt{\frac{\sin x \cos\alpha + \cos x \sin\alpha}{\sin x}} + C$$

$$= \frac{-2}{\sin\alpha}\sqrt{\frac{\sin(x+\alpha)}{\sin x}} + C$$

Question 19:

Integrate $\dfrac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}, x \in [0,1]$

Solution 19:

Let $I = \displaystyle\int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}} dx$

It is known that, $\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x} = \dfrac{\pi}{2}$

$$\Rightarrow I=\int\frac{\left(\frac{\pi}{2}-\cos^{-1}\sqrt{x}\right)-\cos^{-1}\sqrt{x}}{\frac{\pi}{2}}dx$$

$$=\frac{2}{\pi}\int\left(\frac{\pi}{2}-2\cos^{-1}\sqrt{x}\right)dx$$

$$=\frac{2}{\pi}.\frac{\pi}{2}\int 1.dx-\frac{4}{\pi}\int\cos^{-1}\sqrt{x}dx$$

$$=x-\frac{4}{\pi}\int\cos^{-1}\sqrt{x}dx \qquad ...(1)$$

$$Let\ I_1=\int\cos^{-1}\sqrt{x}dx$$

Also. let $\sqrt{\text{x}}=t\Rightarrow dx=2t\,dt$

$$\Rightarrow I_1=2\int\cos^{-1}t.t\,dt$$

$$=2\left[\cos^{-1}t.\frac{t^2}{2}-\int\frac{-1}{\sqrt{1-t^2}}.\frac{t^2}{2}dt\right]$$

$$=t^2\cos^{-1}t+\int\frac{t^2}{\sqrt{1-t^2}}dt$$

$$=t^2\cos^{-1}t-\int\frac{1-t^2-1}{\sqrt{1-t^2}}dt$$

$$=t^2\cos^{-1}t-\int\sqrt{1-t^2}dt+\int\frac{1}{\sqrt{1-t^2}}dt$$

$$=t^2\cos^{-1}t-\frac{1}{2}\sqrt{1-t^2}-\frac{1}{2}\sin^{-1}t+\sin^{-1}t$$

$$=t^2\cos^{-1}t-\frac{1}{2}\sqrt{1-t^2}+\frac{1}{2}\sin^{-1}t$$

From equation (1), we obtain

$$I=x-\frac{4}{\pi}\left[t^2\cos^{-1}t-\frac{t}{2}\sqrt{1-t^2}+\frac{1}{2}\sin^{-1}t\right]$$

$$=x-\frac{4}{\pi}\left[x\cos^{-1}\sqrt{x}-\frac{\sqrt{x}}{2}\sqrt{1-x}+\frac{1}{2}\sin^{-1}\sqrt{x}\right]$$

$$=x-\frac{4}{\pi}\left[x\left(\frac{\pi}{2}-\sin^{-1}\sqrt{x}\right)-\frac{\sqrt{x-x^2}}{2}+\frac{\pi}{2}\sin^{-1}\sqrt{x}\right]$$

$$=x-2x+\frac{4x}{\pi}\sin^{-1}\sqrt{x}+\frac{2}{\pi}\sqrt{x-x^2}-\frac{2}{\pi}\sin^{-1}\sqrt{x}$$

$$=-x+\frac{2}{\pi}\left[(2x-1)\sin^{-1}\sqrt{x}\right]+\frac{2}{\pi}\sqrt{x-x^2}+C$$

$$=\frac{2(2x-1)}{\pi}\sin^{-1}\sqrt{x}+\frac{2}{\pi}\sqrt{x-x^2}-x+C$$

Question 20:

Integrate $\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$

Solution 20:

$$I=\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}dx$$

Let $x=\cos^2\theta \Rightarrow dx=-2\sin\theta\cos\theta\, d\theta$

$$I=\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\left(-2\sin\theta\cos\theta\right)d\theta$$

$$=-\int\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\sin 2\theta\, d\theta$$

$$=-\int\tan\frac{\theta}{2}.2\sin\theta\cos\theta\, d\theta$$

$$=-2\int\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)\cos\theta d\theta$$

$$=-4\int\sin^2\frac{\theta}{2}\cos\theta d\theta$$

$$=-4\int\sin^2\frac{\theta}{2}.\left(2\cos^2\frac{\theta}{2}-1\right)d\theta$$

$$=-4\int\left(2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\right)d\theta$$

$$=-8\int\sin^2\frac{\theta}{2}.\cos^2\frac{\theta}{2}d\theta+4\int\sin^2\frac{\theta}{2}d\theta$$

$$=-2\int\sin^2\theta\, d\theta+4\int\sin^2\frac{\theta}{2}d\theta$$

$$=-2\int\left(\frac{1-\cos 2\theta}{2}\right)d\theta+4\int\frac{1-\cos\theta}{2}d\theta$$

$$=-2\left[\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right]+4\left[\frac{\theta}{2}-\frac{\sin\theta}{2}\right]+C$$

$$=-\theta+\frac{\sin 2\theta}{2}+2\theta-2\sin\theta+C$$

$$=\theta+\frac{\sin 2\theta}{2}+2\sin\theta+C$$

$$=\theta+\frac{2\sin\theta\cos\theta}{2}-2\sin\theta+C$$

$$=\theta+\sqrt{1-\cos^2\theta}.\cos\theta-2\sqrt{1-\cos^2\theta}+C$$
$$=\cos^{-1}\sqrt{x}+\sqrt{1-x}.\sqrt{x}-2\sqrt{1-x}+C$$
$$=-2\sqrt{1-x}+\cos^{-1}\sqrt{x}+\sqrt{x(1-x)}+C$$
$$=-2\sqrt{1-x}+\cos^{-1}\sqrt{x}+\sqrt{x-x^2}+C$$

Question 21:

Integrate $\frac{2+\sin 2x}{1+\cos 2x}e^x$

Solution 21:

$$I=\int\left(\frac{2+\sin 2x}{1+\cos 2x}\right)e^x$$
$$=\int\left(\frac{2+2\sin x\cos x}{2\cos^2 x}\right)e^x$$
$$=\int\left(\frac{1+\sin x\cos x}{\cos^2 x}\right)e^x$$
$$=\int\left(\sec^2 x+\tan x\right)e^x$$
$$Let\ \ f(x)=\tan x\Rightarrow f'(x)=\sec^2 x$$
$$\therefore I=\int\left(f(x)+f'(x)\right)e^x dx$$
$$=e^x f(x)+C$$
$$=e^x\tan x+C$$

Question 22:

Integrate $\frac{x^2+x+1}{(x+1)^2(x+2)}$

Solution 22:

Let $\frac{x^2+x+1}{(x+1)^2(x+2)}=\frac{A}{(x+1)}+\frac{B}{(x+1)^2}+\frac{C}{(x+2)}$ ...(1)

$$\Rightarrow x^2+x+1=A(x+1)(x+2)+B(x+2)+C\left(x^2+2x+1\right)$$
$$\Rightarrow x^2+x+1=A\left(x^2+3x+2\right)+B(x+2)+C\left(x^2+2x+1\right)$$
$$\Rightarrow x^2+x+1=(A+C)x^2+(3A+B+2C)x+(2A+2B+C)$$

Equating the coefficients of x^2, x and constant term, we obtain

A + C = 1

3A + B + 2C = 1

2A + 2B + C = 1

On solving these equations, we obtain
A = −2, B = 1, and C = 3
From equation (1), we obtain

$$\frac{x^2+x+1}{(x+1)^2(x+2)}=\frac{-2}{(x+1)}+\frac{3}{(x+2)}+\frac{1}{(x+1)^2}$$

$$\int\frac{x^2+x+1}{(x+1)^2(x+2)}dx=-2\int\frac{1}{x+1}dx+3\int\frac{1}{(x+2)}dx+\int\frac{1}{(x+1)^2}dx$$

$$=-2\log|x+1|+3\log|x+2|-\frac{1}{(x+1)}+C$$

Question 23:

Integrate $\tan^{-1}\sqrt{\frac{1-x}{1+x}}$

Solution 23:

I = $\tan^{-1}\sqrt{\frac{1-x}{1+x}}$

Let $x=\cos\theta\Rightarrow dx=-\sin\theta d\theta$

$$I=\int\tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}-(\sin\theta d\theta)$$

$$=-\int\tan^{-1}\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\sin\theta d\theta$$

$$=-\int\tan^{-1}\tan\frac{\theta}{2}.\sin\theta d\theta$$

$$=-\frac{1}{2}\int\theta.\sin\theta d\theta$$

$$=-\frac{1}{2}\left[\theta.(-\cos\theta)-\int 1.(-\cos\theta)d\theta\right]$$

$$=-\frac{1}{2}[-\theta\cos\theta+\sin\theta]$$

$$=+\frac{1}{2}\theta\cos\theta-\frac{1}{2}\sin\theta$$

$$=\frac{1}{2}\cos^{-1}x.x-\frac{1}{2}\sqrt{1-x^2}+C$$

$$=\frac{x}{2}\cos^{-1}-\frac{1}{2}\sqrt{1-x^2}+C$$

$$=\frac{1}{2}\left(x\cos^{-1}x-\sqrt{1-x^2}\right)+C$$

Question 24:

Integrate $\dfrac{\sqrt{x^2+1}\left[\log\left(x^2+1\right)-2\log x\right]}{x^4}$

Solution 24:

$$\frac{\sqrt{x^2+1}\left[\log\left(x^2+1\right)-2\log x\right]}{x^4}=\frac{\sqrt{x^2+1}}{x^4}\left[\log\left(x^2+1\right)-\log x^2\right]$$

$$=\frac{\sqrt{x^2+1}}{x^4}\left[\log\left(\frac{x^2+1}{x^2}\right)\right]$$

$$=\frac{\sqrt{x^2+1}}{x^4}\log\left(1+\frac{1}{x^2}\right)$$

$$=\frac{1}{x^3}\sqrt{\frac{x^2+1}{x^2}}\log\left(1+\frac{1}{x^2}\right)$$

$$=\frac{1}{x^3}\sqrt{1+\frac{1}{x^2}}\log\left(1+\frac{1}{x^2}\right)$$

$$Let\ 1+\frac{1}{x^2}=t\Rightarrow\frac{-2}{x^3}dx=dt$$

$$\therefore I=\int\frac{1}{x^3}\sqrt{1+\frac{1}{x^2}}\log\left(1+\frac{1}{x^2}\right)dx$$

$$=-\frac{1}{2}\int\sqrt{t}\log t dt$$

$$=-\frac{1}{2}\int t^{\frac{1}{2}}.\log t dt$$

Integrating by parts, we obtain

$$I=-\frac{1}{2}\left[\log t.\int t^{\frac{1}{2}}.dt-\left\{\left(\frac{d}{dt}\log t\right)\int t^{\frac{1}{2}}dt\right\}dt\right]$$

$$=-\frac{1}{2}\left[\log t.\frac{t^{\frac{3}{2}}}{\frac{3}{2}}-\int\frac{1}{t}.\frac{t^{\frac{3}{2}}}{\frac{3}{2}}dt\right]$$

$$=-\frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\log t-\frac{2}{3}\int t^{\frac{1}{2}}dt\right]$$

$$=-\frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\log t-\frac{4}{9}t^{\frac{3}{2}}\right]$$

$$=-\frac{1}{3}t^{\frac{3}{2}}\log t+\frac{2}{9}t^{\frac{3}{2}}$$

$$=-\frac{1}{3}t^{\frac{3}{2}}\left[\log t-\frac{2}{3}\right]$$

$$=-\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{\frac{3}{2}}\left[\log\left(1+\frac{1}{x^2}\right)-\frac{2}{3}\right]+C$$

Question 25:

$$\int_{\frac{\pi}{2}}^{\pi} e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$$

Solution 25:

$$I=\int_{\frac{\pi}{2}}^{\pi} e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$$

$$=\int_{\frac{\pi}{2}}^{\pi} e^x\left(\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right)dx$$

$$=\int_{\frac{\pi}{2}}^{\pi}\left(\frac{\cos ec^2\frac{x}{2}}{2}-\cot\frac{x}{2}\right)dx$$

$$Let\ \ f(x)=-\cot\frac{x}{2}$$

$$\Rightarrow f'(x)=-\left(-\frac{1}{2}\cos ec^2\frac{x}{2}\right)=\frac{1}{2}\cos ec^2\frac{x}{2}$$

$$\therefore I=\int_{\frac{\pi}{2}}^{\pi} e^x\left(f(x)+f'(x)\right)dx$$

$$=\left[e^x.f(x)dx\right]_{\frac{\pi}{2}}^{\pi}$$

$$=-\left[e^x.\cot\frac{x}{2}\right]_{\frac{\pi}{2}}^{\pi}$$

$$=-\left[e^{\pi}.\cot\frac{\pi}{2}-e^{\frac{\pi}{2}}.\cot\frac{\pi}{4}\right]$$

$$=-\left[e^{\pi}.0-e^{\frac{\pi}{2}}.1\right]$$

$$=e^{\frac{\pi}{2}}$$

Question 26:

$$\int_0^{\frac{\pi}{4}}\frac{\sin x\cos x}{\cos^4 x+\sin^4 x}dx$$

Solution 26:

Let $I=\int_0^{\frac{\pi}{4}}\frac{\sin x\cos x}{\cos^4 x+\sin^4 x}dx$

$\Rightarrow I=\int_0^{\frac{\pi}{4}}\frac{\frac{(\sin x\cos x)}{\cos^4 x}}{\frac{\left(\cos^4 x+\sin^4 x\right)}{\cos^4 x}}dx$

$\Rightarrow I=\int_0^{\frac{\pi}{4}}\frac{\tan x\sec^2 x}{1+\tan^4 x}dx$

Let $\tan^2 x=t\Rightarrow 2\tan x\sec^2 xdx=dt$

When x = 0, t = 0 and when $x=\frac{\pi}{4},t=1$

$\therefore I=\frac{1}{2}\int_0^1\frac{dt}{1+t^2}$

$=\frac{1}{2}\left[\tan^{-1}t\right]_0^1$

$=\frac{1}{2}\left[\tan^{-1}1\tan^{-1}0\right]$

$=\frac{1}{2}\left[\frac{\pi}{4}\right]$

$=\frac{\pi}{8}$

Question 27:

$\int_0^{\frac{\pi}{2}}\frac{\cos^2 xdx}{\cos^2 x+4\sin^2 x}$

Solution 27:

Let $I=\int_0^{\frac{\pi}{2}}\frac{\cos^2 x}{\cos^2 x+4\sin^2 x}dx$

$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\cos^2 x}{\cos^2 x+4\left(1-\cos^2 x\right)}dx$

$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\cos^2 x}{\cos^2 x+4-4\cos^2 x}dx$

$\Rightarrow I=\frac{-1}{3}\int_0^{\frac{\pi}{2}}\frac{4-3\cos^2 x}{\cos^2 x+4-4\cos^2 x}dx$

$\Rightarrow I=\frac{-1}{3}\int_0^{\frac{\pi}{2}}\frac{4-3\cos^2 x}{4-3\cos^2 x}dx+\frac{1}{3}\int_0^{\frac{\pi}{2}}\frac{4}{4-3\cos^2 x}dx$

$\Rightarrow I=\frac{-1}{3}\int_0^{\frac{\pi}{2}}1dx+\frac{1}{3}\int_0^{\frac{\pi}{2}}\frac{4\sec^2 x}{4\sec^2 x-3}dx$

$$\Rightarrow I = \frac{-1}{3}[x]_0^{\frac{\pi}{2}} + \frac{1}{3}\int_0^{\frac{\pi}{2}} \frac{4\sec^2 x}{4(1+\tan^2 x)-3}dx$$

$$\Rightarrow I = -\frac{\pi}{6} + \frac{2}{3}\int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1+4\tan^2 x}dx \qquad ...(1)$$

Consider, $\int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1+4\tan^2 x}dx$

Let 2 tanx = t $\Rightarrow$ 2sec^{2}xdx = dt

When x = 0, t = 0 and when x = $\frac{\pi}{2}$, t = ∞

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1+4\tan^2 x}dx = \int_0^{\infty} \frac{dt}{1+t^2}$$

$$= \left[\tan^{-1} t\right]_0^{\infty}$$

$$= \left[\tan^{-1}(\infty) - \tan^{-1}(0)\right]$$

$$= \frac{\pi}{2}$$

Therefore, from (1), we obtain

$$I = -\frac{\pi}{6} + \frac{2}{3}\left[\frac{\pi}{2}\right] = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

Question 28:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$$

Solution 28:

Let $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$

$$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x + \cos x)}{\sqrt{-(-\sin 2x)}} dx$$

$$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{-(-1+1-2\sin\cos x)}} dx$$

$$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x + \cos x)}{\sqrt{1-(\sin^2 x\cos^2 x - 2\sin x\cos x)}} dx$$

$$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x + \cos x)dx}{\sqrt{1-(\sin x - \cos x)^2}}$$

Let $(\sin x - \cos x) = t = (\sin x + \cos x)dx = dt$

when $x=\dfrac{\pi}{6}, t=\left(\dfrac{1-\sqrt{3}}{2}\right)$ and when x = $\dfrac{\pi}{3}, t=\left(\dfrac{\sqrt{3}-1}{2}\right)$

$$I=\int_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$$

$$\Rightarrow I=\int_{-\left(\frac{1-\sqrt{3}}{2}\right)}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$$

As $\dfrac{1}{\sqrt{1-(-t)^2}}=\dfrac{1}{\sqrt{1-t^2}}$, therefore, $\dfrac{1}{\sqrt{1-t^2}}$ is an even function.

$$\int_{-a}^{a} f(x)dx=2\int_0^a f(x)dx$$

It is known that if f(x) is an even function, then

$$\Rightarrow I=2\int_0^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$$

$$=\left[2\sin^{-1}t\right]_0^{\frac{\sqrt{3}-1}{2}}$$

$$=2\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right) = 2(\pi/12) = \pi/6$$

Question 29:

$$\int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}}$$

Solution 29:

Let $I=\int_0^1 \dfrac{dx}{\sqrt{1+x}-\sqrt{x}}$

$$I=\int_0^1 \frac{1}{\left(\sqrt{1+x}-\sqrt{x}\right)}\times\frac{\left(\sqrt{1+x}+\sqrt{x}\right)}{\left(\sqrt{1+x}+\sqrt{x}\right)}dx$$

$$=\int_0^1 \frac{\left(\sqrt{1+x}+\sqrt{x}\right)}{1+x-x}dx$$

$$=\int_0^1 \sqrt{1+x}dx+\int_0^1\sqrt{x}dx$$

$$=\left[\frac{2}{3}(1+x)^{\frac{2}{3}}\right]_0^1\left[\frac{2}{3}(x)^{\frac{3}{2}}\right]_0^1$$

$$=\frac{2}{3}\left[(2)^{\frac{2}{3}}-1\right]+\frac{2}{3}[1]$$

$$=\frac{2}{3}(2)^{\frac{2}{3}}$$

$=\frac{2.2\sqrt{2}}{3}$

$=\frac{4\sqrt{2}}{3}$

Question 30:

$\int_0^{\frac{\pi}{4}} \frac{\sin x+\cos x}{9+16\sin 2x}dx$

Solution 30:

Let I = $\int_0^{\frac{\pi}{4}} \frac{\sin x+\cos x}{9+16\sin 2x}dx$

Also let sin x – cos x = t $\Rightarrow$ (cos x + sinx) dx = dt

When x = 0, t = - 1 and when x = $\frac{\pi}{4}$, t = 0

$\Rightarrow (\sin x-\cos x)^2 = t^2$

$\Rightarrow \sin^2+\cos^2-2\sin x\cos x = t^2$

$\Rightarrow 1-\sin 2x = t^2$

$\Rightarrow \sin 2x = 1-t^2$

$\therefore I = \int_{-1}^{0} \frac{dt}{9+16(1-t^2)}$

$= \int_{-1}^{0} \frac{dt}{9+16-16t^2}$

$= \int_{-1}^{0} \frac{dt}{25-16t^2} = \int_{-1}^{0} \frac{dt}{(5)^2-(4t)^2}$

$= \frac{1}{4}\left[\frac{1}{2(5)}\log\left|\frac{5+4t}{5-4t}\right|\right]_{-1}^{0}$

$= \frac{1}{40}\left[\log(1)-\log\left|\frac{1}{9}\right|\right]$

$= \frac{1}{40}\log 9$

Question 31:

$\int_0^{\frac{\pi}{2}} \sin 2x\tan^{-1}(\sin x)dx$

Solution 31:

Let $I = \int_0^{\frac{\pi}{2}} \sin 2x\tan^{-1}(\sin x)dx = \int_0^{\frac{\pi}{2}} 2\sin x\cos x\tan^{-1}(\sin x)dx$

Also, let sin x = t $\Rightarrow$ cos x dx = dt

When x = 0, t = 0 and when x = $\frac{\pi}{2}$, t = 1

$\Rightarrow I = 2\int_0^1 t\tan^{-1}(t)\,dt$ … (1)

Consider $\int t.\tan^{-1} tdt = \tan^{-1} t.\int tdt - \int\left\{\frac{d}{dt}\left(\tan^{-1} t\right)\int tdt\right\}dt$

$= \tan^{-1} t.\frac{t^2}{2} - \int \frac{1}{1+t^2}.\frac{t^2}{2}dt$

$= \frac{t^2\tan^{-1} t}{2} - \frac{1}{2}\int \frac{t^2+1-1}{1+t^2}dt$

$= \frac{t^2\tan^{-1} t}{2} - \frac{1}{2}\int 1.dt + \frac{1}{2}\int \frac{1}{1+t^2}dt$

$= \frac{t^2\tan^{-1} t}{2} - \frac{1}{2}.t + \frac{1}{2}\tan^{-1} t$

$\Rightarrow \int_0^1 t.\tan^{-1} tdt = \left[\frac{t^2\tan^{-1} t}{2} - \frac{t}{2} + \frac{1}{2}\tan^{-1} t\right]_0^1$

$= \frac{1}{2}\left[\frac{\pi}{4} - 1 + \frac{\pi}{4}\right]$

$= \frac{1}{2}\left[\frac{\pi}{2} - 1\right] = \frac{\pi}{4} - \frac{1}{2}$

From equation (1), we obtain

$I = 2\left[\frac{\pi}{4} - \frac{1}{2}\right] = \frac{\pi}{2} - 1$

Question 32:

$\int_0^{\pi} \frac{x\tan x}{\sec x + \tan x}dx$

Solution 32:

Let $\int_0^{\pi} \frac{x\tan x}{\sec x + \tan x}dx$ ...(1)

$I = \int_0^{\pi}\left\{\frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)}\right\}dx$ $\left(\int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$

$\Rightarrow I = \int_0^{\pi}\left\{\frac{-(\pi - x)\tan x}{-(\sec x + \tan x)}\right\}dx$

$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)\tan x}{\sec x + \tan x}dx$ ...(2)

Adding (1) and (2), we obtain

$$2I = \int_0^{\pi} \frac{\pi \tan x}{\sec x + \tan x} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x} + \frac{\sin x}{\cos x}} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{\sin x + 1 - 1}{1 + \sin x} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} 1.dx - \pi \int_0^{\pi} \frac{1}{1 + \sin x} dx$$

$$\Rightarrow 2I = \pi [x]_0^{\pi} - \pi \int_0^{\pi} \frac{1 - \sin x}{\cos^2 x} dx$$

$$\Rightarrow 2I = \pi^2 - \pi \int_0^{\pi} \left(\sec^2 x - \tan x \sec x\right) dx$$

$$\Rightarrow 2I = \pi^2 - \pi \left[\tan x - \sec x\right]_0^{\pi}$$

$$\Rightarrow 2I = \pi^2 - \pi \left[\tan \pi - \sec \pi - \tan 0 + \sec 0\right]$$

$$\Rightarrow 2I = \pi^2 - \pi \left[0 - (-1) - 0 + 1\right]$$

$$\Rightarrow 2I = \pi^2 - 2\pi$$

$$\Rightarrow 2I = \pi(\pi - 2)$$

$$\Rightarrow I = \frac{\pi}{2}(\pi - 2)$$

Question 33:

$$\int_1^4 \left[|x-1| + |x-2| + |x-3|\right] dx$$

Solution 33:

Let $I = \int_1^4 \left[|x-1| + |x-2| + |x-3|\right] dx$

$$\Rightarrow I = \int_1^4 |x-1|\, dx + \int_1^4 |x-2|\, dx + \int_1^4 |x-3|\, dx$$

$$I = I_1 + I_2 + I_3 \qquad \ldots(1)$$

where, $I_1 = \int_1^4 |x-1|\, dx$, $I_2 = \int_1^4 |x-2|\, dx$, *and* $I_3 = \int_1^4 |x-3|\, dx$

$$I_1 = \int_1^4 |x-1|\, dx$$

$(x-1) \geq 0$ *for* $1 \leq x \leq 4$

$$\therefore I_1 = \int_1^4 (x-1)\, dx$$

$$\Rightarrow I_1 = \left[\frac{x^2}{2} - x\right]_1^4$$

$$\Rightarrow I_1 = \left[8-4-\frac{1}{2}+1\right] = \frac{9}{2} \qquad ...(2)$$

$$I_2 = \int_1^4 |x-2|\,dx$$

$$x-2 \geq 0 \text{ for } 2 \leq x \leq 4 \text{ and } x-2 \leq 0 \text{ for } 1 \leq x \leq 2$$

$$\therefore I_2 = \int_1^2 (2-x)\,dx + \int_2^4 (x-2)\,dx$$

$$\Rightarrow I_2 = \left[2x-\frac{x^2}{2}\right]_1^2 + \left[\frac{x^2}{2}-2x\right]_2^4$$

$$\Rightarrow I_2 = \left[4-2-2+\frac{1}{2}\right] + [8-8-2+4]$$

$$\Rightarrow I_2 = \frac{1}{2}+2 = \frac{5}{2} \qquad ...(3)$$

$$I_3 = \int_1^4 |x-3|\,dx$$

$$x-3 \geq 0 \text{ for } 3 \leq x \leq 4 \text{ and } x-3 \leq 0 \text{ for } 1 \leq x \leq 3$$

$$\therefore I_3 = \int_1^3 (3-x)\,dx + \int_3^4 (x-3)\,dx$$

$$\Rightarrow I_3 = \left[3x-\frac{x^2}{2}\right]_1^3 + \left[\frac{x^2}{2}-3x\right]_3^4$$

$$\Rightarrow I_3 = \left[9-\frac{9}{2}-3+\frac{1}{2}\right] + \left[8-12-\frac{9}{2}+9\right]$$

$$\Rightarrow I_3 = [6-4] + \left[\frac{1}{2}\right] = \frac{5}{2} \qquad ...(4)$$

From equations (1), (2), (3), and (4), we obtain

$$I = \frac{9}{2}+\frac{5}{2}+\frac{5}{2} = \frac{19}{2}$$

Question 34:

Prove $\int_1^3 \frac{dx}{x^2(x+1)} = \frac{2}{3} + \log\frac{2}{3}$

Solution 34:

Let $I = \int_1^3 \frac{dx}{x^2(x+1)}$

Also, let $\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$

$$\Rightarrow 1 = Ax(x+1) + B(x+1) + C(x^2)$$

$$\Rightarrow 1 = Ax^2 + Ax + Bx + B + Cx^2$$

Equating the coefficients of x^2, x, and constant term, we obtain

A + C = 0

A + B = 0
B = 1
On solving these equations, we obtain
A = −1, C = 1, and B = 1

$$\therefore \frac{1}{x^2(x+1)} = \frac{-1}{x} + \frac{1}{x^2} + \frac{1}{(x+1)}$$

$$\Rightarrow I = \int_1^3 \left\{ -\frac{1}{x} + \frac{1}{x^2} + \frac{1}{(x+1)} \right\} dx$$

$$= \left[-\log x - \frac{1}{x} + \log(x+1) \right]_1^3$$

$$= \left[\log\left(\frac{x+1}{x} \right) - \frac{1}{x} \right]_1^3$$

$$= \log\left(\frac{4}{3} \right) - \frac{1}{3} - \log\left(\frac{2}{1} \right) + 1$$

$$= \log 4 - \log 3 - \log 2 + \frac{2}{3}$$

$$= \log 2 - \log 3 + \frac{2}{3}$$

$$= \log\left(\frac{2}{3} \right) + \frac{2}{3}$$

Hence, the given result is proved.

Question 35:

Prove $\int_0^4 xe^x dx = 1$

Solution 35:

Let $I = \int_0^4 xe^x dx$

Integrating by parts, we obtain

$$I = x\int_0^4 e^x dx - \int_0^1 \left\{ \left(\frac{d}{dx}(x) \right) \int e^x dx \right\} dx$$

$$= \left[xe^x \right]_0^1 - \int_0^1 e^x dx$$

$$= \left[xe^x \right]_0^1 - \left[e^x \right]_0^1$$

$$= e - e + 1$$

$$= 1$$

Hence, the given result is proved.

Question 36:

Prove $\int_{-1}^{1} x^{17}\cos^4 x dx = 0$

Solution 36:

Let $I = \int_{-1}^{1} x^{17}\cos^4 x dx$

Also, let $f(x) = x^{17}\cos^4 x$

$\Rightarrow f(-x) = (-x)^{17}\cos^4(-x) = -x^{17}\cos^4 x = -f(x)$

Therefore, f (x) is an odd function.

It is known that if f(x) is an odd function, then $\int_{-a}^{a} f(x) dx = 0$

$\therefore I = \int_{-1}^{1} x^{17}\cos^4 x dx = 0$

Hence, the given result is proved.

Question 37:

Prove $\int_{0}^{\frac{\pi}{2}} \sin^3 x dx = \frac{2}{3}$

Solution 37:

Let $I = \int_{0}^{\frac{\pi}{2}} \sin^3 x dx$

$I = \int_{0}^{\frac{\pi}{2}} \sin^2 x.\sin x dx$

$= \int_{0}^{\frac{\pi}{2}} (1-\cos^2 x)\sin x dx$

$= \int_{0}^{\frac{\pi}{2}} \sin x dx - \int_{0}^{\frac{\pi}{2}} \cos^2 x.\sin x dx$

$= [-\cos x]_{0}^{\frac{\pi}{2}} + \left[\frac{\cos^3 x}{3}\right]_{0}^{\frac{\pi}{2}}$

$= 1 + \frac{1}{3}[-1] = 1 - \frac{1}{3} = \frac{2}{3}$

Hence, the given result is proved.

Question 38:

Prove $\int_{0}^{\frac{\pi}{4}} 2\tan^3 x dx = 1 - \log 2$

Solution 38:

Let $I = \int_{0}^{\frac{\pi}{4}} 2\tan^3 x dx$

$$I=\int_0^{\frac{\pi}{4}} 2\tan^2 x\tan x dx = 2\int_0^{\frac{\pi}{4}}\left(\sec^2 x-1\right)\tan x dx$$

$$=2\int_0^{\frac{\pi}{4}}\sec^2 x\tan x dx - 2\int_0^{\frac{\pi}{4}}\tan x dx$$

$$=2\left[\frac{\tan^2 x}{2}\right]_0^{\frac{\pi}{4}} + 2\left[\log\cos x\right]_0^{\frac{\pi}{4}}$$

$$=1+2\left[\log\cos\frac{\pi}{4}-\log\cos 0\right]$$

$$=1+2\left[\log\frac{1}{\sqrt{2}}-\log 1\right]$$

$$=1-\log 2-\log 1=1-\log 2$$

Hence, the given result is proved.

Question 39:

Prove $\int_0^1 \sin^{-1} x dx = \frac{\pi}{2}-1$

Solution 39:

Let $\int_0^1 \sin^{-1} x dx$

$$\Rightarrow I=\int_0^1 \sin^{-1} x.1.dx$$

Integrating by parts, we obtain

$$I=\left[\sin^{-1} x.x\right]_0^1-\int_0^1\frac{1}{\sqrt{1-x^2}}.xdx$$

$$=\left[x\sin^{-1} x\right]_0^1+\frac{1}{2}\int_0^1\frac{(-2x)}{\sqrt{1-x^2}}dx$$

Let $1-x^2=t \Rightarrow -2x\,dx = dt$

When x = 0, t = 1 and when x = 1, t = 0

$$I=\left[x\sin^{-1} x\right]_0^1+\frac{1}{2}\int_0^1\frac{dt}{\sqrt{t}}$$

$$=\left[x\sin^{-1} x\right]_0^1+\frac{1}{2}\left[2\sqrt{t}\right]_1^0$$

$$=\sin^{-1}(1)+\left[-\sqrt{1}\right]$$

$$=\frac{\pi}{2}-1$$

Hence, the given result is proved.

Question 40:

Evaluate $\int_0^1 e^{2-3x}dx$ as a limit of a sum.

Solution 40:

Let I = $\int_0^1 e^{2-3x}dx$

It is known that,

$$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+...+f(a+(n-1)h)\right]$$

where, $h = \dfrac{b-a}{n}$

Here, $a=0,\ b=1$, and $f(x)=e^{2-3x}$

$$\Rightarrow h=\frac{1-0}{n}=\frac{1}{n}$$

$$\therefore \int_0^1 e^{2-3x}dx = (1-0)\lim_{n\to\infty}\frac{1}{n}\left[f(0)+f(0+h)+...+f(0+(n-1)h)\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[e^2+e^{2-3x}+...+e^{2-3(n-1)h}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[e^2\left\{1+e^{-3h}+e^{-6h}+e^{-9h}+...+e^{-3(n-1)h}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[e^2\left\{\frac{1-\left(e^{-3h}\right)^n}{1-\left(e^{-3h}\right)}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[e^2\left\{\frac{1-e^{-\frac{3}{n}\times n}}{1-e^{-\frac{3}{n}}}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[\frac{e^2\left(1-e^{-3}\right)}{1-e^{-\frac{3}{n}}}\right]$$

$$=e^2\left(e^{-3}-1\right)\lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{e^{-\frac{3}{n}}-1}\right]$$

$$=e^2\left(e^{-3}-1\right)\lim_{n\to\infty}\left(-\frac{1}{3}\right)\left[\frac{-\frac{3}{n}}{e^{-\frac{3}{n}}-1}\right]$$

$$=\frac{e^2\left(e^{-3}-1\right)}{3}\lim_{n\to\infty}\left[\frac{-\frac{3}{n}}{e^{-\frac{3}{n}}-1}\right]$$

$$=\frac{-e^2\left(e^{-3}-1\right)}{3}(1) \qquad \left[\lim_{n\to\infty}\frac{x}{e^x-1}\right]$$

$$=\frac{-e^{-1}+e^{2}}{3}$$
$$=\frac{1}{3}\left(e^{2}-\frac{1}{e}\right)$$

Question 41:

$\int \frac{dx}{e^{x}+e^{-x}}$ is equal to

A. $\tan^{-1}\left(e^{x}\right)+C$

B. $\tan^{-1}\left(e^{-x}\right)+C$

C. $\log\left(e^{x}-e^{-x}\right)+C$

D. $\log\left(e^{x}+e^{-x}\right)+C$

Solution 41:

Let $I=\int \frac{dx}{e^{x}+e^{-x}}dx=\int \frac{e^{x}}{e^{2x}+1}dx$

Also, let $e^{x}=t \Rightarrow e^{x}dx=dt$

$$\therefore I=\int \frac{dt}{1+t^{2}}$$
$$=\tan^{-1}t+C$$
$$=\tan^{-1}\left(e^{x}\right)+C$$

Hence, the correct Answer is A.

Question 42:

$\int \frac{\cos 2x}{\left(\sin x+\cos x\right)^{2}}dx$ is

A. $\frac{-1}{\sin x+\cos x}+C$

B. $\log\left|\sin x+\cos x\right|+C$

C. $\log\left|\sin x-\cos x\right|+C$

D. $\frac{1}{\left(\sin x+\cos x\right)^{2}}+C$ equal to

Solution 42:

Let $I=\int \frac{\cos 2x}{\left(\sin x+\cos x\right)^{2}}dx$

$$I=\int \frac{\cos^2 x-\sin^2 x}{(\sin x+\cos x)^2}dx$$
$$=\int \frac{(\cos x+\sin x)(\cos x-\sin x)}{(\sin x+\cos x)^2}dx$$
$$=\int \frac{\cos x-\sin x}{\cos x+\sin x}dx$$
Let $\cos x+\sin x=t \Rightarrow (\cos x-\sin x)dx=dt$
$$\therefore I=\int \frac{dt}{t}$$
$$=\log|t|+C$$
$$=\log|\cos x+\sin x|+C$$
Hence, the correct Answer is B.

Question 43:

If f (a + b – x) = f (x), then $\int_a^b xf(x)dx$ is equal to

A. $\frac{a+b}{2}\int_a^b f(b-x)dx$

B. $\frac{a+b}{2}\int_a^b f(b+x)dx$

C. $\frac{b-a}{2}\int_a^b f(x)dx$

D. $\frac{a+b}{2}\int_a^b f(x)dx$

Solution 43:

Let I = $\int_a^b xf(x)dx$ … (1)

$$I=\int_a^b (a+b-x)f(a+b-x)dx \qquad \left(\int_a^b f(x)dx=\int_a^b f(a+b-x)dx\right)$$
$$\Rightarrow I=\int_a^b (a+b-x)f(x)dx$$
$$\Rightarrow I=(a+b)\int_a^b f(x)dx-I \qquad [\text{using}(1)]$$
$$\Rightarrow I+I=(a+b)\int_a^b f(x)dx$$
$$\Rightarrow 2I=(a+b)\int_a^b f(x)dx$$
$$\Rightarrow I=\left(\frac{a+b}{2}\right)\int_a^b f(x)dx$$
Hence, the correct Answer is D.

Question 44:

The value of $\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx$ is

A. 1
B. 0
C. − 1
D. $\frac{\pi}{4}$

Solution 44:

Let $I = \int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx$

$\Rightarrow I = \int_0^1 \tan^{-1}\left(\frac{x-(1-x}{1+x(1-x)}\right)dx$

$\Rightarrow I = \int_0^1 \left[\tan^{-1} x - \tan^{-1}(1-x)\right]dx \qquad ...(1)$

$\Rightarrow I = \int_0^1 \left[\tan^{-1}(1-x) - \tan^{-1}(1-1+x)\right]dx$

$\Rightarrow I = \int_0^1 \left[\tan^{-1}(1-x) - \tan^{-1}(x)\right]dx$

$\Rightarrow I = \int_0^1 \left[\tan^{-1}(1-x) - \tan^{-1}(x)\right]dx \qquad ...(2)$

Adding (1) and (2), we obtain

$\Rightarrow 2I = \int_0^1 \left(\tan^{-1} x - \tan^{-1}(1-x) - \tan^{-1}(1-x) - \tan^{-1} x\right)dx$

$\Rightarrow 2I = 0$

$\Rightarrow I = 0$

Hence, the correct Answer is B.